Dynamic Modeling of the Correlation Smile

Alfred Hamerle and Christian Scherr

Abstract We discuss the equity-based pricing of CDX tranches within a structural dynamic approach and focus on the valuation impact of general model specifications. Therefore, we examine the influence of market dynamics, idiosyncratic jumps, loss term structures and portfolio heterogeneity on the pricing of tranches. The resulting spread deviations are quantified through implied correlations because this scales premium payments across all tranches to a comparable level and, in addition, enables reliable inferences on the meaning of the discussed model features.

JEL Classification: G12, G13, G15
Keywords: CDX index tranches, correlation smile, Kou model, relative pricing

1 Introduction

The recent debate on the relative pricing of equity and credit risk markets (see [7, 8, 9, 14, 16]) raises the issue of the extent to which the applied models themselves drive the published results. In particular, this emerges all the more with respect to the large variety of proposed models and corresponding findings. An initial way to address this topic seems to be a comparison of different valuation techniques by referring to a homogenous set of input data. However, this in fact fails because even within a certain class of model type the number of parameters and model components turns out to be significantly different. Concerning structural approaches, one might deal

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for example with static models, comprising only a sparse number of parameters (see e.g. [8]), or adopt fully dynamic techniques with dozens of variables as in [7].

Because of these differences, we restrict ourselves to a structural dynamic approach and examine the impact of general model specifications on the pricing of credit derivatives, such as the inclusion of idiosyncratic jumps. In this sense, we proceed similarly to [1], who quantify the effects of ignoring empirical regulatories on the valuation of CDO tranches. Their aim, however, is different because they wish to explain the appearance of the so-called correlation smile (see also [2, 11, 17]), which proves the poor reliability of the standard one-factor Gaussian copula model. In addition, all proposed techniques are of static nature, whereas our analysis refers to a basic approach that already captures the most important empirical phenomena and thus serves as a reference for measuring the impact of general model specifications.

To set up the basic approach, we adopt the structural model recently proposed by [10]. Using CAPM-like techniques, they introduce a simple dynamic model to overcome the main disadvantages associated with purely diffusion-based techniques. In addition to a component that depicts continuous changes, they also include jumps to capture discontinuous information. Hence, our basic model contains the most important characteristics that, according to [7], a reliable approach should offer. Firstly, it is intended to be fully dynamic, which is accomplished by definition because we are dealing with a time-continuous stochastic process. Secondly, the model must not be exclusively based on a diffusion motion because this leads to the so-called predictability of default, and thus short-time spreads become vanishingly low (see e.g. [19]). Due to the presence of jumps, our approach is not in danger of exhibiting these disadvantages.

To quantify the impact of different model specifications, we compare the corresponding risk premiums to those of our basic approach. However, the spread rates of different tranches are generally of a different scale, and thus, if measured in absolute values, slight deviations in the equity tranche acquire much more weight than large deviations within the senior tranches. To avoid such effects, we adopt the concept of implied correlations because, as a consequence, quotes are of the same magnitude and spread deviations become comparable. Thus, we evaluate the deviations with respect to our basic model and report the pricing effect of model changes in terms of implied correlations.

The proposed model changes are chosen in such a way as to preserve the analytical tractability of the different approaches. For example, we add idiosyncratic jumps to the asset value process. Analogously to the idiosyncratic diffusion motion, these depict changes in firm value that are not influenced by the macroeconomic dynamics but reflect information causing discontinuous movements. A crucial topic within our analysis is the weight we assign to these idiosyncratic jumps because this directly influences the magnitude of correlation among the assets in the modeled reference pool. Correlation matters, because it affects the terminal loss distribution of the portfolio, which in turn influences tranche prices. For example, if there is a significant number of scenarios in which the portfolio loss is close to zero, the equity tranche can survive, at least in part. Hence, the spread rates of equity tranches decrease. For senior tranches, things are different. Increasing the probability of extreme losses entails the eventuality of subordinated capital being wiped out completely and also senior tranches getting hit. Because spread rates reflect expected losses, premium payments have to increase. A decreasing correlation reduces the incidence of extreme events and the loss distribution becomes more centered. As a consequence, equity tranches often suffer substantial losses and have to offer high spread payments. Conversely, senior tranches are hit sparsely and thus only have to yield low premiums on the notional.

However, if the correlation were the only quantity determining tranche prices, dynamic models would not yield significant advantages in the context of modeling credit derivatives because terminal distributions are also specified by proposing static models. Yet, static models have a tremendous disadvantage: they cannot describe the evolution of portfolio loss dynamics over time. Yet, these are also essential to evaluate the loss dynamics of tranches. The temporal growth of tranche losses affects the spread rate of a tranche because spread payments always refer to the remaining notional. If tranches are likely to suffer early losses, spread rates have to rise in return for missed payments. Senior tranches are expected to have very low losses, and therefore the explicit loss dynamics should not significantly influence the associated premiums. This changes, however, as one moves through the capital structure down to the equity tranche. Due to its position, this exhibits maximum sensitivity to early defaults in the portfolio. This motivates our quantitative analysis, which determines the extent to which loss dynamics in the underlying portfolio influence tranche prices.

Besides idiosyncratic jumps and loss dynamics, there are two more topics we wish to discuss in the course of this paper, namely the meaning of market return dynamics and the homogeneity assumption. Whereas there is no doubt about the influence of equity dynamics, a clear economic theory on the impact of the homogeneity assumption is missing. Therefore, our empirical analysis is also intended to yield new insights into this topic.

Accordingly, the remainder of the paper is organized as follows. In Section 2, we provide a brief overview of credit derivatives and some details on the correlation smile. The mathematics of the market as well as the asset value dynamics are discussed in Section 3. In the context of the model analysis presented in Section 4, we quantify the impacts of the proposed model changes. A conclusion is given in Section 5.

2 Credit derivatives and correlation smile

2.1 Credit derivatives

2.1.1 CDS indices

Analogous to equity indices, comprising a certain number of stocks, CDS indices represent a portfolio of credit default swap contracts. In the empirical section of this article, we focus on the CDX North American Investment Grade index (CDX.NA.IG), which aggregates 125 equally weighted CDS contracts, each written on a North American investment grade name. There are several maturities of this index, namely 1, 2, 3, 4, 5, 7 and 10 years, whereby the contract with the five-year horizon offers the highest degree of liquidity. The CDX.NA.IG is revised every six months on March 20 and September 20, the so-called roll dates. On these dates, both defaulted as well as illiquid names are replaced. Similar to a CDS contract, the issuer (protection buyer) has to pay quarterly spread premiums to the investor (protection seller). In the case of default, the latter is obliged to render compensation for the loss caused by the defaulted company. In general, this loss, also referred to as Loss Given Default (LGD), is a firm-specific, stochastic variable. For reasons of simplicity, here we fix the LGD to the standard value of 0.6. As a further consequence of default, the notional value of the contract is reduced by a factor of $\frac{1}{125}$, disregarding the actual loss. In a risk-neutral environment, the spread rate of this contract is given by

$$s^i := \frac{LGD \cdot \sum_{i=1}^{n} \sum_{j=1}^{m} e^{-rt_j} \cdot \mathbb{P}\left(t_{j-1} < \tau_i \leq t_j\right)}{\sum_{i=1}^{n} \sum_{j=1}^{m} \Delta_j \cdot e^{-rt_j} \cdot \mathbb{P}\left(\tau_i > t_j\right)} \tag{1}$$

Here, $\Delta_j := t_j - t_{j-1}$ denotes the time period between two subsequent payment dates, r the risk-free interest rate and τ_i the default time of reference name i.

2.1.2 Index tranches

By dividing their capital structure, CDS indices are also used to create structured finance securities, called index tranches. These tranches induce a vertical capital structure on the index and are specified by the covered loss range. A tranche begins to suffer losses as the portfolio loss L_t exceeds the attachment point α, and its notional is completely wiped out if the portfolio loss increases beyond the detachment point β. For example, the CDX.NA.IG has the tranches 0-3% (equity), 3-7% (mezzanine), 7–10%, 10–15%, 15–30% (senior) and 30–100% (super-senior). The spread rate of a tranche is given by

$$s_{\alpha,\beta} := \frac{\sum_{j=1}^{m} e^{-rt_j} \cdot \left[\mathbb{E}\left(L_{\alpha,\beta}^{t_j}\right) - \mathbb{E}\left(L_{\alpha,\beta}^{t_{j-1}}\right)\right]}{\sum_{j=1}^{m} \Delta_j \cdot e^{-rt_j} \cdot \left[1 - \mathbb{E}\left(L_{\alpha,\beta}^{t_j}\right)\right]} \tag{2}$$

where the loss profile of a tranche follows

$$L_{\alpha,\beta}^{t} := \frac{\min(\beta, L_t) - \min(\alpha, L_t)}{\beta - \alpha} \tag{3}$$

2.2 Correlation smiles

In the context of modeling credit derivatives, the one-factor Gaussian copula model is similar to the Black-Scholes approach for the pricing of options. Hence, it does not come as a surprise that there is also a phenomenon, called the correlation smile, that corresponds to the empirically observed volatility smile.

2.2.1 Volatility Smile

The famous Black-Scholes pricing formula owes its popularity mainly to the fact that, based on the intuitive Brownian motion, [4] elaborated an analytical formula for the pricing of European options, including the contemporary stock price S_0, the strike level K, the maturity T, the interest rate r and the volatility σ of the underlying asset. Whereas S_0, K, T and r are explicitly observable quantities or parameters characterizing the proposed contract, the volatility can, at best, be estimated. In turn, only the volatility parameter is available to control the results within the Black-Scholes model. Given the market price of a completely specified European option, one can fit the Black-Scholes model to this quote by choosing the (unique) volatility that yields the desired value. If the Black-Scholes model could completely describe market dynamics, all the (implied) volatilities would be identical across different maturities and strike levels. Yet, these volatilities are not generally constant but yield patterns that resemble smiles or skews if plotted against the strike level or maturity. This suggests that the Black-Scholes model is not suited to replicate option prices. However, the general popularity of this model is testified by the fact that it is market convention to quote option prices in terms of implied volatility. This fictive number, placed in the “wrong” Black-Scholes formula, by construction reveals the predefined value and therefore offers an alternative way to report prices of options.

2.2.2 Correlation Smile

Within the Gaussian model, there are only two parameters that can be used to control the model's features, namely the default barrier $\tilde{D}$ and the homogenous asset return correlation ρ. It is a general convention to fix the default barrier such that the model spread matches the empirically observed index spread. As a consequence, ρ is the only parameter affecting tranche prices, and the market spread of a fixed tranche is replicated by evaluating the level of the generic or implied correlation that yields this spread. For a given set of tranche prices on an arbitrary day, this procedure is expected to reveal five different correlations.[1] The resulting confliction can be resolved simply by realizing that the one-factor Gaussian copula model does not offer a reliable description of the pooled assets (see e.g. [20]). However, analogous to the Black-Scholes model, the Gaussian approach also offers an analytical formula for the valuation of tranches[2], which in turn explains its popularity and the fact that tranche spreads are also quoted in terms of implied correlations.

3 Asset value dynamics

3.1 General model features

With respect to our basic asset pool model, we specify the firm value dynamics to satisfy the stochastic differential equation stated by [12]:

$$\frac{dA(t)}{A(t-)} = (r - \lambda_a \zeta_a)\, dt + \sigma_a dB_a(t) + d\left[\sum_{i=1}^{N_a^m(t)} (V_{a,i} - 1)\right] \tag{4}$$

Hence, three basic components control the evolution of a company's asset value return: the drift component, the diffusion motion and the jump part. The drift rate is specified by $(r - \lambda_a \zeta_a)$, which contains the risk-free interest rate as well as the compensator that accounts for the expected drift caused by the jump process. Continuously occurring changes are depicted by the Brownian diffusion $\sigma_a B_a(t)$. The jump part specifies systematic jumps to which all companies are exposed. The number of these jumps is denoted by $N_a^m(t)$ and follows a Poisson process with the intensity λ_a. The random number $V_{a,i},\ i \in \{1, ..., N_a^m(t)\}$, is characterized by the density of its logarithmic version

$$Y_{a,i} := \ln(V_{a,i}) \tag{5}$$

that follows an asymmetric double exponential distribution:

$$f_{Y_{a,i}}(y) = p \cdot \eta_1 e^{-\eta_1 y} \mathbf{1}_{y \geq 0} + q \cdot \eta_2 e^{\eta_2 y} \mathbf{1}_{y<0}, \quad \eta_1 > 1,\, \eta_2 > 0 \tag{6}$$

[1] Super-senior tranches of the pre-crisis CDX.NA.IG are commonly assumed to be (almost) riskless and thus omitted from our analysis.

[2] For technical details, we refer interested readers to [18].

Therefore, $p,q \geq 0$, $p+q=1$, define the conditional probabilities of upward and downward jumps. Because $N_a^m(t)$ and $V_{a,i}$ are stochastically independent, the process

$$C_a^m(t) := \sum_{i=1}^{N_a^m(t)} (V_{a,i}-1) \tag{7}$$

is a compound Poisson process, with expectation

$$\mathbb{E}\left[C_a^m(t)\right] = \lambda_a t \left(\frac{p\eta_1}{\eta_1-1} + \frac{q\eta_2}{\eta_2+1} - 1\right) \tag{8}$$

Performing calculations in the context of exponential Lévy models, one generally refers to logarithmic returns because these can be treated more easily. Applying Itô's Lemma to

$$X(t) := \ln\left[A(t)\right] \tag{9}$$

yields

$$X(t) = \left(r - \frac{\sigma_a^2}{2} - \lambda_a \zeta_a\right) t + \sigma_a B_a(t) + \sum_{i=1}^{N_a^m(t)} Y_{a,i} \tag{10}$$

Without loss of generality, we assume $A_0 = 0$, and hence the logarithmic return $X(t)$ is given by a standard Lévy process that comprises continuous as well as discontinuous movements.

3.2 First passage time distribution

In modeling credit risk, dynamic approaches are usually specified as first passage time models. This concept was introduced by [3] and accounts for the fact that a company can default at any time during the credit period. A default is triggered the moment the asset value touches or crosses some predefined default boundary, which represents the company's level of liabilities. The first passage time τ is defined mathematically as follows:

$$\tau := \inf\{t | A_t \leq D\} = \inf\{t | X_t \leq b\} \tag{11}$$

Here, D denotes the default barrier and b its logarithmic version. Because in our model setting the loss dynamics are determined solely by the default dynamics, the distribution of the first passage time, according to (2), is crucial.

There are only a few types of processes that offer an analytically known distribution of τ. For example, this pertains to the standard Brownian motion and spectrally negative Lévy processes. The Kou model applied in this paper also features an analytically known distribution of the first passage time, as formulated by [13] and [15]. For a comprehensive summary of the (technical) details, we refer interested readers to [18].

The analytical nature of the proposed first passage time model enables a very fast (numerical) determination of loss dynamics and, based on these, the company's spread rate. In turn, given a quoted spread rate, the calibration of a homogenous pool can be conducted by a numerical optimization algorithm, due to the linearity of the expectation operator. If there were no analytically known distribution, calibration would have to be done by simulation techniques, which, despite the rapid growth of computational power, are still very time-consuming and also may potentially yield biased results. This especially appears over the course of extended time periods as well as processes with jumps ([6, 13]). Therefore, the analyticity of our modeling approach, enabling unbiased and fast evaluations at firm and portfolio level, constitutes a major advantage of the presented approach.

3.3 Integration of market risk

3.3.1 Modeling equity dynamics

Besides analytical knowledge about the first passage time distribution, there is another important feature of the Kou model, namely the closed-form option-pricing formula. Extending the classical Black-Scholes approach, [12] calculated an explicit pricing function for European options where the underlying equity dynamics are given by

$$\frac{dS(t)}{S(t-)} = (r - \lambda_s \zeta_s)\, dt + \sigma_s dB_s(t) + d\left[\sum_{i=1}^{N_s(t)} (V_{s,i} - 1)\right] \tag{12}$$

Analogous to the asset value model, the random number $V_{s,i}$, $i \in \{1, ..., N_s(t)\}$, is characterized by the density of its logarithmic version

$$Y_{s,i} := \ln(V_{s,i}) \tag{13}$$

that also exhibits an asymmetric double exponential distribution:

$$f_{Y_{s,i}}(y) = p \cdot \xi_1 e^{-\xi_1 y} \mathbf{1}_{y \geq 0} + q \cdot \xi_2 e^{\xi_2 y} \mathbf{1}_{y < 0}, \quad \xi_1 > 1, \xi_2 > 0 \tag{14}$$

Hence, the price $C(K,T)$ of a European call option written on an equity asset that follows (12) can be evaluated as a function of the strike level K and the maturity T:[3]

$$\begin{aligned} C(K,T) = \Upsilon &\left(r + \frac{1}{2}\sigma_s^2 - \lambda_s \zeta_s, \sigma_s, \tilde{\lambda}_s, \tilde{p}, \tilde{\xi}_1, \tilde{\xi}_2; \ln(K), T\right) \\ &- K \exp(-rT) \cdot \Upsilon \left(r - \frac{1}{2}\sigma_s^2 - \lambda_s \zeta_s, \sigma_s, \lambda_s, p, \xi_1, \xi_2; \ln(K), T\right) \end{aligned} \tag{15}$$

[3] The explicit functional dependence is stated in [18].

where

$$\tilde{p} = \frac{p}{1+\zeta_s} \cdot \frac{\xi_1}{\xi_1 - 1}, \quad \tilde{\xi}_1 = \xi_1 - 1, \quad \tilde{\xi}_2 = \xi_2 + 1, \quad \tilde{\lambda}_s = \lambda_s\,(\zeta_s + 1) \tag{16}$$

If we specify $S(t)$ to reflect the dynamics of an equity index, for example the S&P 500, and use this index as a proxy for the market dynamics, the logarithmic market returns are also given by a double exponential jump-diffusion process. By calibrating these equity dynamics to market data, given a fixed σ_s, we can ascertain the unknown parameter values of the jump part. Following [10], we choose $\sigma_s = 0.1$.

3.3.2 Coupling equity and asset dynamics

In our asset value model, the diffusion parameter σ_a, as well as λ_a, are used to set up the coupling between market and asset dynamics. This coupling reflects the notion that companies are exposed to both market and idiosyncratic risks. Whereas market risk simultaneously influences the evolution of all companies in a portfolio, idiosyncratic risks independently affect firm values. Adopting this basic idea of the CAP-model, we specify the asset value diffusion to follow the market diffusion up to a factor β. Additionally, we introduce an independent Brownian motion B_a^i to depict the continuous evolution of idiosyncratic risk. Thus, the asset value diffusion is given by

$$\sigma_a B_a = \beta \sigma_s B_s + \sigma_a^i B_a^i \sim \mathcal{N}\left(0, \sigma_a^2\right) \tag{17}$$

Here, we made use of the fact that the superposition of independent Brownian motions again turns out to be Brownian.

With respect to the jump part of our firm value model, we apply the parameters λ_s and ξ_s to specify the corresponding asset value dynamics. Due to the fact that jumps in the firm value are caused exclusively by jumps in the equity process, we fix the jump rate λ_a to be equal to λ_s. However, the jump distribution must be different because within our approach the level of debt is assumed to be constant, which in turn reduces the effects of discontinuous movements in the market value. We account for this fact by adopting the β factor introduced above and define

$$Y_{a,i} := \beta \cdot Y_{s,i} \tag{18}$$

Applying the transformation formula, it is easy to show that this way of proceeding preserves the distribution characteristic and thus proves to be consistent with the given firm value dynamics. Furthermore, the distribution parameter of $Y_{a,i}$ can be evaluated as follows:

$$\eta_a = \frac{1}{\beta}\xi_s \tag{19}$$

reflecting, on average, the damped amplitude of jumps. For reasons of simplicity, we restrict ourselves to the limiting case of $q = 1$, concerning both asset and equity dynamics. Thus, we define $\eta_a := \eta_2$ and $\xi_s := \xi_2$.

4 Model changes and correlation smiles

4.1 Data description

The database for our analysis relies primarily on quotes that were offered in addition to the publication of [8] and are available at the webpage of the publishing journal. These quotes comprise data on five-year S&P 500 index options, spread rates of the five-year CDX.NA.IG and associated tranches as well as time-congruent swap rates. The swap rates are also offered by www.swap-rates.com and used as risk-free interest rates. The time series cover the period from September 22, 2004 to September 19, 2007, which corresponds exactly to the duration period of the CDX.NA.IG Series 3 through Series 8. In addition, the data on S&P 500 index options provide daily information on option prices with respect to 13 different strike levels and also report the time series of the S&P 500 index level.

4.2 Basic model

For the purpose of calibrating our basic model, we utilize prices of S&P 500 index options and spread rates of the five-year CDX.NA.IG that were observed on February 6, 2006. We choose this date because within our analysis we wish to analyze the pricing impact of model changes with respect to a common market environment.[4] On average, the pre-crisis spread rate of the five-year CDX.NA.IG can be calculated to about 45 bps (the exact mean value amounts to 45.87 bps), which, for example, was the market quote on February 6, 2006. In addition, this date is also located in the center of our time series.

To calibrate our market model, we must back out the optimal value of (λ_s, ξ_s). Because all the other input variables required for the pricing of options are known, namely the contemporary index level, strike price, interest rate and maturity, we perform a numerical optimization procedure that minimizes the sum of in-sample quadratic pricing errors:

$$\mathscr{E}(\lambda_s, \xi_s) := \sum_{i=1}^{13} \left[\tilde{P}_i(\lambda_s, \xi_s) - P_i\right]^2 \tag{20}$$

where $\tilde{P}_i$ denotes the model price and P_i the corresponding empirical value. As a result of this procedure, we obtain

$$(\lambda_s, \xi_s)_{opt} := (0.125, 2.91) \tag{21}$$

which is used to determine the model implied volatility skew shown by the solid line in Figure 1. This curve, as well as the market-implied volatilities, marked by

[4] According to [7] and [8], we specify Series 3 through 8 to represent the pre-crisis period.

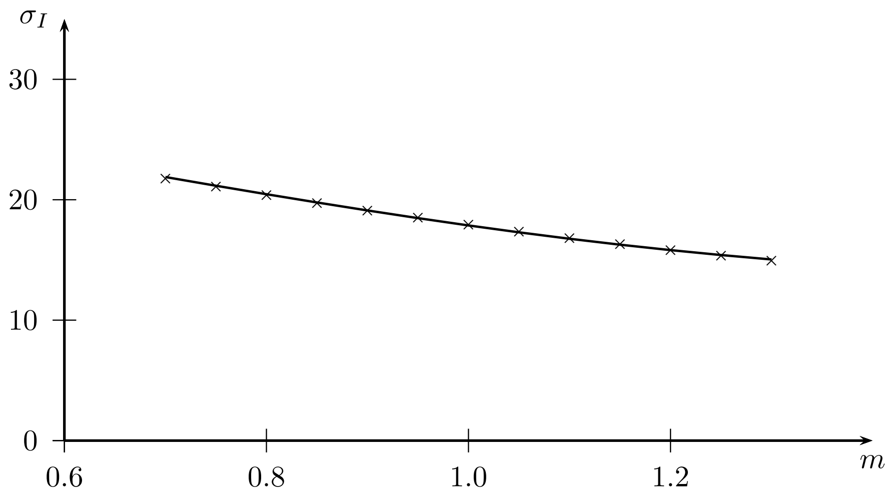

Fig. 1: **Implied volatility σ_I of the market model**
The solid line shows the resulting function extracted from five-year S&P 500 index option prices (marked by the crosses). All values are quoted in percent.

the crosses, are plotted against the moneyness level m, which we define by

$$m := \frac{K}{S_0} \tag{22}$$

On average, the relative pricing error amounts to 0.30%, which emphasizes the high fitting quality of the chosen market model, relying only on two degrees of freedom. Concerning the pool model, we choose $\beta = 0.5$ and $\sigma_a = 0.2$ to capture the main results of a corresponding survey performed by [7]. In this regard, the sparse number of parameters constitutes a further advantage of our approach because besides $(\lambda_s, \xi_s)_{opt}$ we only have to determine the logarithmic default boundary b. This can be done by evaluating the (unique) zero of

$$s_i^m(b) - s_i^e \tag{23}$$

where s_i^m denotes the model implied spread rate and s_i^e the empirically observed index spread of the CDX.NA.IG on February 6, 2006. A simple numerical procedure yields $b = -1.141$, which completes our setup.

4.3 Market dynamics

The basic concept of our pricing model refers to the notion that the common dynamics of asset values are affected only by the temporal evolution of the corresponding equity market. In this context, predefined changes in the market dynamics are intended to have a similar impact on the model implied spread rates. If, for example, the risk neutral probability of negative market states increases, premium payments on senior tranches are supposed to rise. By contrast, if option prices imply a significant incidence of positive market states, equity spreads are expected to fall.

Here, we analyze the pricing impact of market dynamics by adopting the couples of jump parameters that imply the minimum and maximum as well as the 25%, 50% and 75% quantile of the terminal logarithmic return variance in our time series:

$$\mathbb{V}\left[\ln\left(S_T\right)\right] = \sigma_s^2 T + 2\frac{\lambda T}{\xi_s^2} \tag{24}$$

We use the resulting parameters to specify our asset value model (besides the default boundary, we keep all the other parameters fixed) and perform a recalibration to match the target value of 45 bps. Accordingly, a further advantage of our modeling approach emerges. Given the numerically determined default barrier, we can prove the reliability of simulated tranche spreads because the applied Monte Carlo techniques must also yield the desired index level. Otherwise, computational efforts have to be increased to avoid biased results. We use the modeled spread rates to back out the implied correlations within the Gaussian model and depict the resulting values in Figure 2.

In addition, Table 1 presents the deviations compared to our basic model. Concerning equity and senior tranches, the extreme specifications of market dynamics impact significantly on the premium payments. In line with the economic mechanism discussed above, the low variance scenario causes equity spreads to rise and senior spreads to fall, whereas the high variance scenario implies reduced payments on equity and heightened payments on senior notionals. These results can be simply explained by the different incidences of both positive as well as negative market states. Due to its position in the capital structure, the mezzanine tranche exhibits only a minimum response to the market dynamics, and consequently spread rates also vary only slightly. In addition, if we focus on the range that comprises the scenarios between the 25% and 75% quantile of the logarithmic return variance, the pricing impact occurs in the economically expected direction, but, surprisingly, also appears to be limited. This finding may potentially be ascribed to the tempered market environment within the pre-crisis period, which causes the corresponding market dynamics to be at a comparable level. However, given our results, a more detailed analysis of the pricing impact of market dynamics would seem to be a worthwhile objective of future research.

4.4 Idiosyncratic jumps

The jump part of our basic model captures solely the arrival of “discontinuous” information, such as political power changes, judicial decisions and so on, which com-

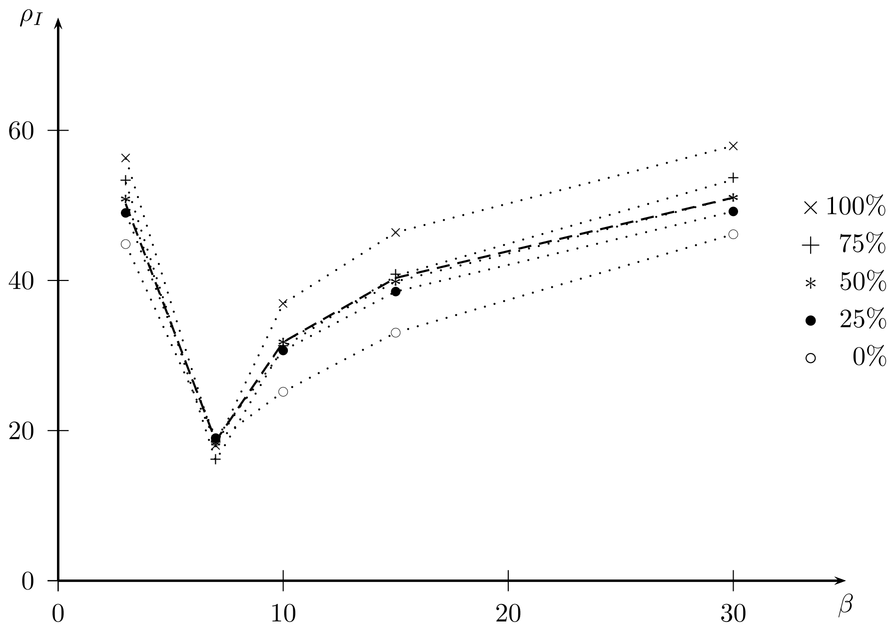

Fig. 2: **Implied correlation ρ_I with respect to changes in market dynamics**
The dashed curve refers to our basic model, whereas the legend symbols specify the different equity dynamics. All values are quoted in percent.

z	0%	25%	50%	75%	100%
ε_1	5.4	1.2	0.6	2.9	6.1
ε_2	0.0	0.4	0.3	2.6	0.6
ε_3	6.6	1.1	0.0	0.7	5.1
ε_4	7.3	1.8	0.5	0.3	6.1
ε_5	4.9	1.8	0.0	2.4	6.9

Table 1: **Deviations of implied correlations caused by the use of different market dynamics**
z symbolizes the various quantiles of the terminal logarithmic return variance and ε_i denotes the deviation of the i-th tranche, where $i = 1$ refers to the equity tranche, $i = 2$ to the mezzanine tranche, etc..

monly affect the modeled asset values. Hence, a more general approach comprises the embedding of idiosyncratic jumps that depict sudden firm-specific events, for example, an unexpected change in the board of directors. Integrating these jumps, of course, entails the mutual dependences of the company dynamics to decline. Consequently, equity spread rates are expected to rise, whereas senior rates are supposed to fall.

To examine these suggestions, we include idiosyncratic jumps by adding the compound Poisson process

$$C_a^i(t) = \sum_{i=1}^{N_a^i(t)} (V_{a,i} - 1) \tag{25}$$

with jump intensity λ_a^i and independent jump variables, whose logarithmic values again follow a double exponential distribution.[5] Furthermore, we choose the jump intensities to follow

$$\begin{aligned} \lambda_a^m &= \rho \cdot \lambda_a \\ \lambda_a^i &= (1-\rho) \cdot \lambda_a, \quad 0 \leq \rho \leq 1 \end{aligned} \tag{26}$$

and define

$$\eta_a^m := \eta_a^i := \eta_a \tag{27}$$

Assuming stochastic independence between the systematic and the idiosyncratic jump part, we obtain, in total, a compound process with jump intensity

$$\rho \cdot \lambda_a + (1-\rho) \cdot \lambda_a = \lambda_a \tag{28}$$

and jump parameter η_a. Hence, in terms of distribution, the jump part of our basic and the present approach are identical. This can easily be seen from the characteristic function of the compound Poisson process $C(t)$:

$$\Phi_{C_t}(u) = \exp\left[\lambda t \int_{\mathbb{R}} \left(e^{iux} - 1\right) f(x) dx\right] \tag{29}$$

where λ denotes the jump intensity and $f(x)$ the density of the jump distribution. Due to the distributional equivalence, the present model does not have to be recalibrated, and one can simply adopt the default boundary of the basic model. In addition, the model implied spread rates of indices with shorter maturities also remain unchanged because within this approach the choice of ρ does not affect the term structure of losses.

In turn, this means that we can calibrate the model to reproduce the quoted index spread, but nevertheless have the flexibility to choose the weighting of jumps. At the limit $\rho = 1$, the proposed model coincides with the basic one, whereas $\rho = 0$ implies that there are no systematic jumps.

To analyze the impact of different levels of ρ, we strobe the interval $[0.8, 0.0]$ by steps of 0.2. The corresponding results are depicted in Figure 3, which in particular shows that across all tranches the choice of ρ crucially affects the implied correlations. Due to the numerical decline of extreme events, equity spreads significantly rise, whereas senior spreads almost vanish. As reported in Table 2, especially the impact on the most senior tranche turns out to be very substantial. In the case of $\rho = 0.2$, as well as $\rho = 0.0$, the Gaussian model cannot reproduce the spread rates of the mezzanine tranche implied by the model and, in addition, a degeneration of the smile pattern can be observed.

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[5] Analogous to our basic model, we restrict ourselves to the limit of almost surely negative jumps.

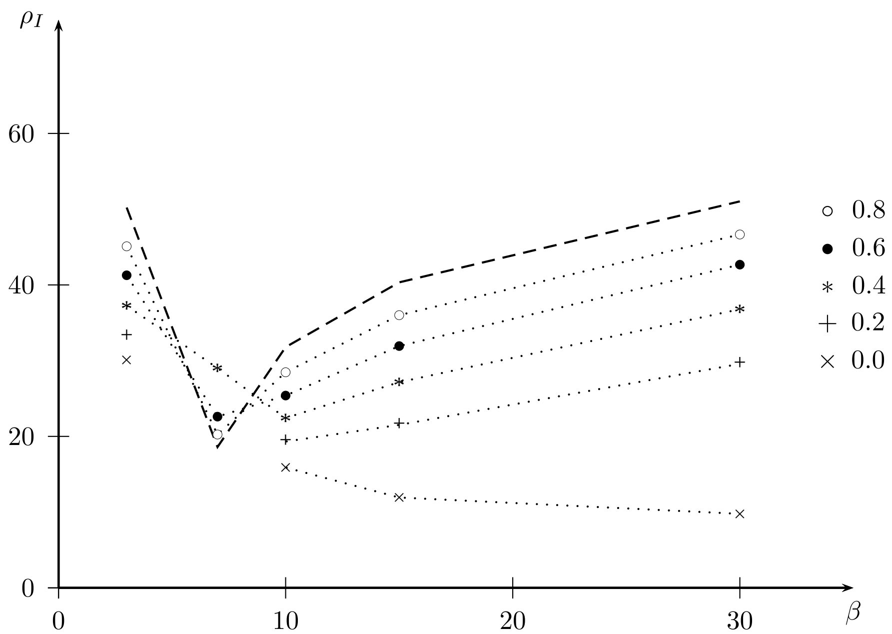

Fig. 3: **Implied correlation ρ_I with respect to the inclusion of idiosyncratic jumps**
The dashed curve refers to the basic model, whereas the legend symbols specify the jump weighting ρ. All values are quoted in percent.

From a general perspective, these results imply that introducing idiosyncratic jumps does not necessarily yield a significant contribution to the term structure properties of a dynamic model but may dramatically influence the pricing of tranches. This finding constitutes the main contribution of our paper, in particular with respect to the contemporary debate on the relative pricing of equity and credit derivatives.

ρ	0.0	0.2	0.4	0.6	0.8
ε_1	20.1	17.1	13.0	8.9	5.1
ε_2	-	-	10.4	4.0	1.7
ε_3	15.9	12.5	9.4	6.4	3.3
ε_4	28.4	18.8	13.2	8.4	4.3
ε_5	41.3	21.5	14.2	8.4	4.4

Table 2: **Deviations of implied correlations caused by the inclusion of idiosyncratic jumps**

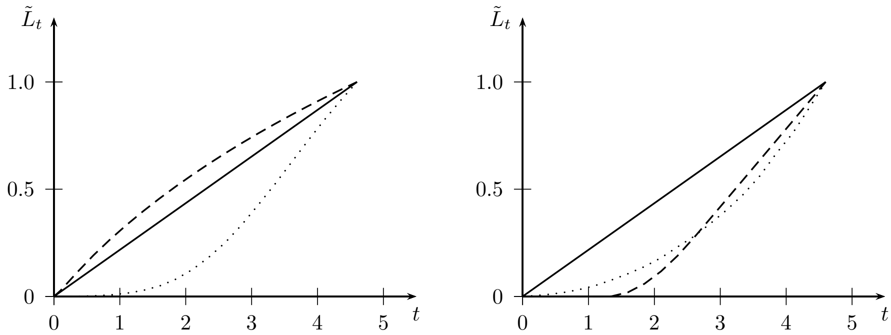

Fig. 4: **Comparison of relative loss term structures $\tilde{L}_t$**
The calculations refer to the one-factor Gaussian copula model (dashed line), the CJS model (solid line) and our basic approach (dotted line) and comprise the equity (left-hand side) and the most senior tranche (right-hand side).

4.5 Term structure of tranche losses

According to the terms of contract, premium payments of tranches always refer to the remaining notional that has not been exhausted as a consequence of portfolio losses. In that regard, due to the absence of loss enhancement, the equity tranche exhibits maximum sensitivity to defaults in the portfolio. For example, if a company declares insolvency soon after contract release, the equity holder immediately loses $\frac{0.6}{125 \cdot 0.03} = 16\%$ of his spread payments. By contrast, senior tranches are expected to suffer very low losses, and thus the explicit loss dynamics should not significantly affect risk premiums. An examination of the impact of loss dynamics is of particular importance with respect to static models because within the modeling process one has to fix generically the corresponding term structures. In the case of the standard Gaussian model, the expected portfolio loss is assumed to grow with a constant hazard rate and thus according to the function

$$\mathbb{E}(L_t) = 1 - e^{-\lambda t} \qquad (30)$$

Here, the hazard rate λ is chosen so that $\mathbb{E}(L_T)$ meets the desired level of loss at maturity.

A further alternative to fixing generically the temporal evolution of losses can be seen from the source code published by [8][6]. Evaluating tranche prices, they assume linearly declining notionals. The term structure implied by our dynamic model is based on the assumption that a company defaults as soon as the asset value touches or deceeds a predefined default barrier. Based on this threshold, the portfolio analysis can be conducted by applying Monte Carlo simulation techniques, whereas

[6] In the following, we use the abbreviation CJS.

the results, among others, are used to determine the term structures of expected losses implied by the model.

To compare the temporal evolution of losses across different tranches, we have to take into account that expected tranche losses are of a different scale. Hence, for each tranche, we rescale the dynamics of losses by the expected loss at maturity and obtain modified loss curves that start at zero, increase monotonically and take one as their terminal value. Figure 4 shows the resulting term structures for the equity and the most senior tranche within the Gaussian, the CJS and our basic approach. Concerning the equity tranche, the one-factor approach shows a “frontloaded” term structure, whereas expected losses of the most senior tranche are “backloaded”. By definition, within the CJS-model, tranche exposures decline linearly over time. The term structures of our basic approach have a similar shape, and both exhibit a convex pattern.

To examine the impact of loss dynamics on the tranche spreads in a general setting, we substitute the first passage time dynamics by

$$L^{\gamma}_{\alpha,\beta}(t) = f_{\gamma}(t) \cdot L^{T}_{\alpha,\beta} \tag{31}$$

where

$$f_{\gamma}(t) := \left(\frac{t}{T}\right)^{\gamma}, \quad \gamma \in \left\{\frac{1}{4}, \frac{1}{2}, 1, 2, 4\right\} \tag{32}$$

γ	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
ε_1	17.8	13.1	7.4	2.0	1.9
ε_2	19.1	9.7	4.8	1.6	0.3
ε_3	6.7	4.9	2.8	1.0	0.3
ε_4	5.4	4.1	2.5	1.0	0.1
ε_5	4.1	3.2	2.1	1.0	0.2

Table 3: **Deviations of implied correlations caused by applying various term structures of tranche losses**

Based on the chosen scenario, denoted by γ, we adopt the terminal tranche losses offered by our basic model and evaluate the spread rates by applying the polynomial term structures. Furthermore, we back out implied correlations and calculate the corresponding deviations to measure the effects on our reference scenario.

Moving from $\gamma = \frac{1}{4}$ to $\gamma = 4$, premium payments decline because losses tend to occur later and, on average, the outstanding notionals are higher. According to Table 3, the sensitivity of the single tranches decreases by moving the capital structure upwards. As economically expected and discussed above, the timing of defaults seriously impacts on the spread rates of equity and mezzanine tranches, whereas senior tranches are less sensitive to the term structure pattern. These findings are also displayed by Figure 5. Hence, the term structures of losses may significantly affect premium payments of tranches, and in particular the generic specification of loss dynamics should be conducted carefully to avoid biased results.

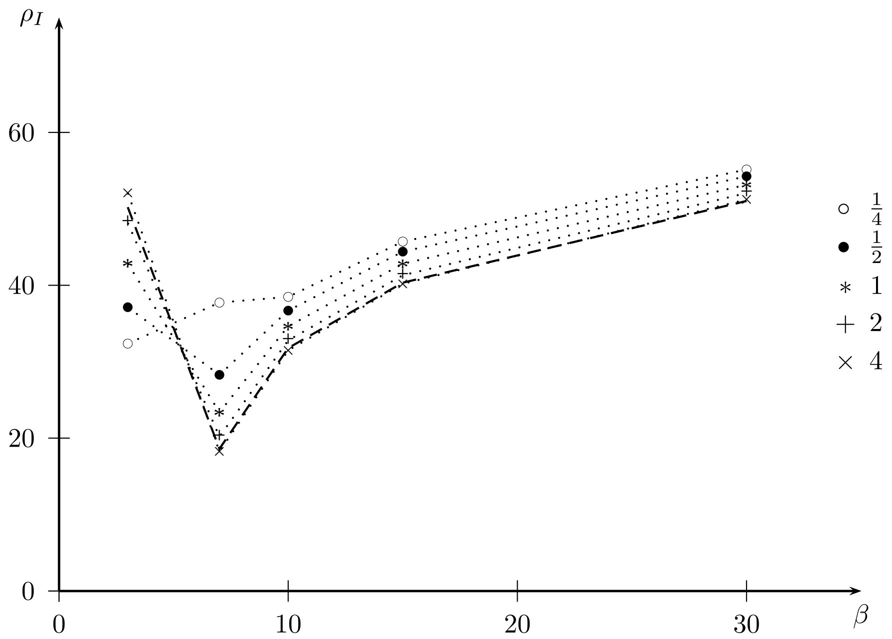

Fig. 5: **Implied correlation ρ_I with respect to the use of different loss dynamics** The dashed curve refers to the basic model, whereas the legend symbols specify the generic loss scenarios. All values are quoted in percent.

4.6 Portfolio heterogeneity

Our basic model refers to a homogenous pool, which implies that under the risk-neutral measure all companies offer identical default dynamics. On the one hand, this assumption is quite a simplification, but on the other hand it also enables an analytical calibration of the portfolio model and thus ensures the approach to be highly applicable. The easiest way to analyze the impact of this assumption is to split the portfolio into two parts that are homogenous by themselves and offer spread rates resembling the observed index spread.

s_1^p	5 bps	15 bps	25 bps	35 bps
ε_1	10.3	5.6	2.2	0.3
ε_2	-	10.8	5.8	1.9
ε_3	10.1	5.3	2.8	1.4
ε_4	4.6	2.5	1.9	1.1
ε_5	3.5	2.6	1.9	1.1

Table 4: **Deviations of implied correlations caused by introducing portfolio heterogeneity**

Here, we fix the homogenous spread rates of $n_1 := 63$ companies to a certain level s_1^p and calculate the corresponding value s_2^p of the remaining $n_2 := 62$ companies. In this context, we follow [5, p. 270], who propose a pricing formula of a CDS index, based purely on the properties of the pooled contracts. Rewriting this formula yields

$$\delta_2 \cdot \left(s_i - s_2^p\right) = \frac{n_1}{n_2} \cdot \delta_1 \cdot \left(s_1^p - s_i\right) \tag{33}$$

The risky duration δ_i, $i = 1,2$, is defined by

$$\delta_i := \sum_{k=1}^{K} e^{-rt_i} \left(1 - p_i^{t_k}\right), \quad t_K = T \tag{34}$$

where

$$p_i^t := \mathbb{P}\left(\tau_i \leq t\right) \tag{35}$$

Given s_1^p, the corresponding default boundary can easily be evaluated. To back out the default boundary of the second part, we use equation (33). Again, due to the analytically known first passage time distribution, we can perform computations very quickly and without bias. The calibration procedure thus yields two different default barriers, which are used to specify the temporal evolution of the portfolio as well as the loss dynamics of tranches. Table 4 reports the corresponding numerical results. The correlation smiles displayed in Figure 6 show a significant impact of portfolio heterogeneity, in particular with respect to the tranches of lower seniority. Figure 6 also shows that an amplification of the portfolio heterogeneity entails a heightened level of implied correlation.

Hence, the homogeneity assumption may imply downward biased spread rates of senior tranches and also cause equity spreads which exceed the actual level. As a consequence, in the context of modeling multi-name derivatives, there should always be a pre-testing of the pooled entities to determine whether or not the homogeneity assumption constitutes a valid simplification.

5 Conclusion

In this article, we analyze the pricing of pre-crisis CDX.NA.IG tranches within a structural dynamic approach. As expected, the mutual dependencies of asset value dynamics, controlled by the weighting of idiosyncratic jumps, affect spread rates at most, whereas the choice of the term structure of losses, as well as the homogeneity assumption, particularly drive tranches of lower seniority. Disregarding portfolio heterogeneity also seems to imply systematically biased results. Surprisingly, our analysis additionally demonstrates a comparatively limited impact of market dynamics on the tranche spreads.

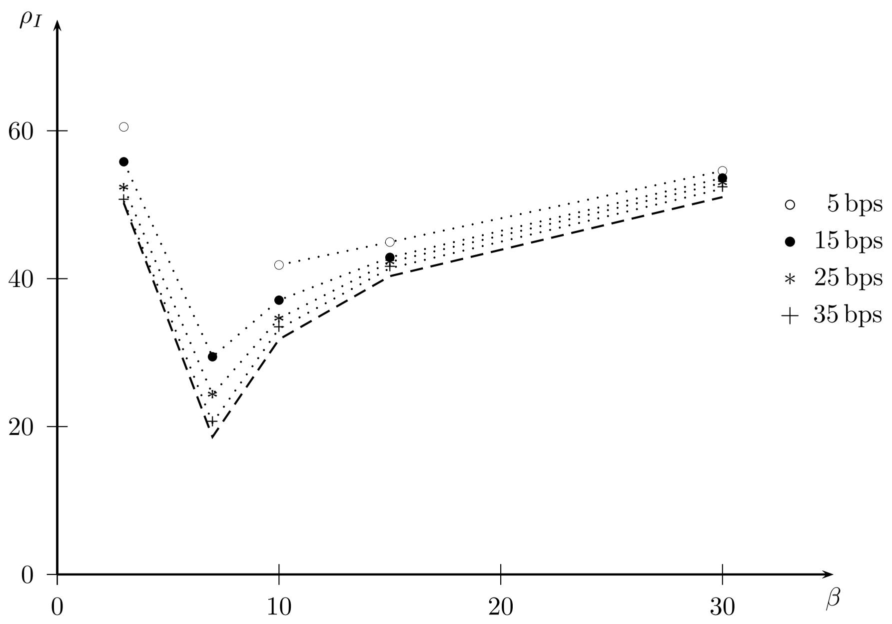

Fig. 6: **Implied correlation ρ_I with respect to the integration of portfolio heterogeneity**
The dashed curve refers to the basic model, whereas the legend symbols specify the different heterogeneity scenarios. All values are quoted in percent.

Of course, there are many issues left that were not covered by our analysis. This is mainly reasoned by the fact that the proposed alterations are chosen in such a way as to ensure analytical tractability, at least at the single-name level. In this regard, further research might, for example, deal with the impact of generalizing model scalars into random variables, which includes recovery rates as well as interest and dividend rates. In addition, the default boundary could be specified as a function of time and the heterogeneity of the pool might be accounted for at a more fine-grained level. However, increasing model complexity always involves the danger of hidden effects emerging, as clearly demonstrated in this article.

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