Correlation Smile, Volatility Skew and Systematic Risk Sensitivity of Tranches

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Abstract

The classical way of treating the correlation smile phenomenon with credit index tranches is to choose a sufficiently flexible model and fit it to tranche market prices. In this article we go a step further and try to explain the tranche prices without directly fitting them. To this end, we use a risk neutral measure of the market factor which we derive from equity index options. The resulting model allows separating the premium for correlation risk from the premium for catastrophe or down-side risk. We show that ignoring the high correlation risk of tranches but allowing for their down-side risk explains the historical market prices fairly well. By contrast, the standard Gaussian copula model allows for the high correlation risk of tranches but disregards the specific down-side risk premia.

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1 Introduction

The motivation of this article is to provide an economic explanation of the correlation smile phenomenon. The correlation smile exists since the beginning of traded index tranches. It is comparable to the volatility smile in the world of options. The problem arises when non-standard, non-traded tranches have to be valued. To calibrate the risk neutral density of the standard Gaussian copula model (Li 2000) it is common to imply the asset correlation from market prices of standard tranches. These calibrated correlations are not unambiguous. They depend on the specific tranche the market price of which one attempts to fit. Different tranches imply different asset correlations.

Many approaches have been suggested which aim at resolving this problem. All of them directly or indirectly aim at a heavier tail of the loss distribution. For example, Burtschell et al. (2005) replace the Gaussian copula by other copulas such as the t-copula or the double-t copula. The increased tail-dependence of these copulas improves the ability of the model to fit all market spreads with a single correlation. Kalemanova et al. (2005) use a normal inverse Gaussian distribution for the market factor in order to generate heavier tails. Moosbrucker (2006) develops a model based on variance-gamma processes. Andersen and Sidenius (2004) as well as Hull et al. (2005) examine a state-contingent correlation parameter. Again heavier tails arise when correlations increase as the market factor becomes more adverse. Krekel (2008) shows that stochastic recovery rates also result in a sufficient fat tail effect to explain market prices. Finally, Agca et al. (2008) study the relevance of a series of assumptions of the Gaussian copula model for the smile phenomenon, such as the Gaussian tails of the asset value process or homogeneity in the pool in terms of asset correlation and recovery rates. They find that the employment of a Gaussian asset value distribution is the most influential assumption contributing to the smile.

A major deficit of most of these approaches is their ad-hoc nature. The primary objective is always to find a risk neutral measure which is consistent with a set of market prices. This is achieved by fitting the model to all available tranche spreads. The resulting measure can be used to price other tranches (“bespoke” tranches). However, this fitting procedure does not lead to an economic explanation of the tranche spreads and therefore lacks theoretical foundation. There is no evidence which of the proposed model extensions are more valid and which are less. In addition, it is not possible to gain an impression of the value of the tranches relative to other risks or markets.

In this article we try to explain market spreads and resolve the correlation smile phenomenon by means of a state-contingent valuation approach. Contrary to the aforementioned ad-hoc procedures we do not fit the model to the market spreads of the tranches. Instead, we derive a risk neutral measure from equity index options. We compare different valuation alternatives which differ in terms of the risk components which are assumed to be price relevant. These are (1) the inclusion of down-side risk premia as they are well known from volatility smile or skew phenomena and (2) the increased systematic risk factor sensitivity of tranches in comparison with corporate bonds.

The first component is a well known phenomenon which can be observed when the Black-Scholes model (Black and Scholes, 1973) is calibrated to option prices. Similar to the correlation smile, market prices of options for different strike prices cannot be explained with a single volatility. The implied volatilities usually show a smile or skew form. A major explanation for this phenomenon is the existence of down-side risk premia, i.e., higher premia for very adverse states of the market (Rubinstein 1994)[1]. The implied volatilities of options on a market index can be used to extract a state price density which includes these down-side risk premia. The procedure is due to Breeden and Litzenberger (1978) and has its theoretical foundation in the state-contingent pricing approach of Arrow (1964) and Debreu (1959). We examine whether these down-side risk premia have explanatory power to resolve the correlation smile.

The second price relevant risk component is systematic risk sensitivity. A basic result of financial economic theory is the relevance of systematic risk. For example, the Capital Asset Pricing Model (CAPM) (Sharpe (1964), Lintner (1965)) states that the risk premia of an asset increase with its correlation to the market portfolio. Since tranches are known to have much higher systematic risk sensitivity than corporate bonds (e.g. Moody's 2008, Coval et al. 2009, Donhauser et al. 2010) we examine whether this adds to the explanation of market prices. Judging by the explanatory power of the different models we find that markets seem to allow for down-side risk but disregard the increased correlation risk of tranches.

The rest of the article is organized as follows. In Section 2 we describe the standard Gaussian copula model and the correlation smile phenomenon. In Section 3 the state-contingent pricing approach is introduced and tested with empirical data. In Section 4 we examine a modified state-contingent pricing approach. Section 5 concludes.

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[1] Rubinstein (1994) uses the term "crash-o-phobia".

2 Standard Gaussian Model and Correlation Smile

The Gaussian copula model has evolved to a standard for synthetic index tranche valuation. To determine the value of a tranche the loss distribution of the underlying pool of credit default swaps has to be calculated.

2.1 The Model

The creditworthiness of any single name $i = 1,...,n$ over the period t is modeled by means of a creditworthiness index comprising a common risk factor M_t as well as an idiosyncratic risk factor $U_{i,t}$:

$$R_{i,t} = \sqrt{\rho_i} \cdot M_t + \sqrt{1-\rho_i} \cdot U_{i,t}$$

Both risk factors as well as $R_{i,t}$ are standard normal. The coefficient ρ_i controls the influence of both factors on the creditworthiness and is a measure of correlation since for any two names *i* and *j* $(i \neq j)$, $Corr(R_{i,t}, R_{j,t}) = \sqrt{\rho_i}\sqrt{\rho_j}$. Obligor *i* defaults if his creditworthiness falls short of some threshold $c_{i,t}$. There is one threshold for any period *t*. The event

$$R_{i,t} < c_{i,t}$$

marks a default over the period *t*.

Let $(t_k)_{k=1}^K = t_1 < t_2 < ... < t_K = T$ denote a discretization of time. The default thresholds are determined as follows. In a first step, given the *T* year credit spread s_i the risk neutral default intensity λ_i is calculated based on the "credit triangle" relation[2], $\lambda_i = s_i / (1 - RR_i)$. Then, the risk neutral default probability until period t_k is given by $p_{i,t_k} = 1 - \exp(-\lambda_i t_k)$. The default threshold is chosen to yield this PD, $\Phi(c_{i,t_k}) = p_{i,t_k} \Leftrightarrow c_{i,t_k} = \Phi^{-1}(p_{i,t_k})$.

Based on this single name model, the losses in the index pool are given by

$$L_{t_k} = \frac{1}{EAD} \sum_{i=1}^{n} EAD_i \cdot (1 - RR_i) \cdot \mathbf{1}_{\{R_{i,t_k} \leq c_{i,t_k}\}} \tag{1}$$

where $EAD = \sum_i EAD_i$ denotes the complete pool exposure at default, EAD_i denotes the exposure of firm *i* and $\mathbf{1}_{\{\}}$ denotes a default indicator.

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[2] The "credit triangle" establishes a very simple relation between spread and default intensity. It is based on the assumption of a constant intensity and continuous compounding.

In the standard Gaussian model the spreads of all names in the pool are assumed to be homogeneous and equal to the index spread[3]. Thus, all obligors have identical risk neutral PDs. Furthermore, the factor loadings as well as the recovery rates are assumed to be constant and homogeneous, i.e., $\rho_i = \rho$ and $RR_i = RR$. Throughout this paper we use a standard recovery rate of $RR = 0.4$. Furthermore, all swaps have equal weight, i.e., $EAD_i = 1/n$.

The capital structure of an index CDO comprises several tranches with strict loss prioritization. Let $0 \leq a < b \leq 1$ denote lower and upper attachment point of a specific tranche (*tr*). Then, the loss of this tranche is given by

$$L_t^{tr} = \frac{1}{b-a}\left(\min\left(L_t, b\right) - \min\left(L_t, a\right)\right) \tag{2}$$

The tranche suffers losses only when the losses in the collateral pool exceed the attachment point *a*. As long as the accumulated loss in the reference portfolio remains below *a,* the tranche remains without loss. When the pool loss exceeds the detachment point *b*, the tranche suffers 100% loss.

The fair value of a tranche implies that its expected discounted loss equals its expected discounted spread income. Thus, the spread of a tranche is given by

$$s^{tr} = \frac{\left(\sum_{k=1}^{K} \exp(-r_f \cdot t_k) \cdot \left(E^Q\left(L_{t_k}^{tr}\right) - E^Q\left(L_{t_{k-1}}^{tr}\right)\right)\right)}{\left(\sum_{k=1}^{K} \Delta t_k \exp(-r_f \cdot t_k) \cdot E^Q(1 - L^{tr}(t_k))\right)}$$

where $t_k = t_1, t_2, ..., t_K = T$ denote quarterly spread payment dates, $\Delta t_k = t_k - t_{k-1}$ and r_f denotes the riskless rate.

Instead of using the model to calculate a tranche spread based on ρ and other parameters, it can be used to determine the implicit correlation which best explains the observed spread of a tranche. This “implied correlation” is not the same for different tranches. It is usually lower for mezzanine tranches and higher for senior and equity tranches. Plotting the correlations in the order of their seniority leads to a curve resembling a smile (see **Figure 1** below).

[3] The index spread is a default-risk weighted average of the single name spreads.

2.2 Empirical Results

In this article we examine the "iTraxx Europe" CDS index and the tranches based on it. The index consists of 125 liquid 5-year CDS contracts on investment grade European firms. The index is updated semi-annually with a new series. We consider three series covering the period from March 2006 to September 2007. The exact start and end dates of each series are given in **Table 1**.

iTraxx	Period
Series 5	21.03.2006 - 20.09.2006
Series 6	21.09.2006 - 20.03.2007
Series 7	21.03.2007 - 20.09.2007

Table 1 *– Start and end date of each examined iTraxx series.*

There are iTraxx tranches with attachment points 0-3, 3-6, 6-9, 9-12, 12-22, and 22-100 percent. Because of its low liquidity we omit the highest tranche (22-100 percent) in our further analysis.

In **Table 2** further descriptive statistics are given. Panel A and B show the means and volatilities of index and tranche spreads. Excluding the high volatility phase of the crisis at the end of series 7, average spreads and spread volatilities were falling from series 5 to series 7. This period was characterized by tightening spreads. Panel C and D show correlations with and without the high volatility phase. The crisis increases the correlations between higher tranches. The effect on subordinate tranches is unclear.

Panel A: Means

	Index	**0 – 3**	**3 – 6**	**6 – 9**	**9 – 12**	**12 – 22**
Series 5	0.003070	0.114969	0.006435	0.001828	0.000739	0.000381
Series 6	0.002468	0.087878	0.005337	0.001489	0.000646	0.000260
Series 7	0.003190	0.101212	0.008496	0.003125	0.001724	0.000952
Series 7*	0.002269	0.078294	0.005104	0.001303	0.000572	0.000253
Total	0.002911	0.101460	0.006758	0.002148	0.001036	0.000531
Total*	0.002665	0.096624	0.005724	0.001585	0.000667	0.000307

Panel B: Standard Deviations

	Index	**0 – 3**	**3 – 6**	**6 – 9**	**9 – 12**	**12 – 22**
Series 5	0.000236	0.010364	0.001116	0.000294	0.000130	0.000056
Series 6	0.000232	0.009185	0.001053	0.000304	0.000148	0.000066
Series 7	0.001218	0.030807	0.004725	0.002623	0.001665	0.001016
Series 7*	0.000146	0.004980	0.000690	0.000207	0.000088	0.000053
Total	0.000792	0.022371	0.003142	0.001683	0.001081	0.000660
Total*	0.000402	0.017789	0.001168	0.000351	0.000145	0.000085

Panel C: Correlations (*all three series*)

	Index	0 – 3	3 – 6	6 – 9	9 – 12	12 – 22
Index	1					
0 – 3	0.903321	1				
3 – 6	0.912238	0.749928	1			
6 – 9	0.920136	0.698677	0.961562	1		
9 – 12	0.899503	0.643117	0.925454	0.987626	1	
12 – 22	0.910653	0.660551	0.892366	0.959713	0.981836	1

Panel D: Correlations (*excluding the high volatility phase*)

	Index	0 – 3	3 – 6	6 – 9	9 – 12	12 – 22
Index	1					
0 – 3	0.995151	1				
3 – 6	0.816644	0.773134	1			
6 – 9	0.859495	0.825022	0.963148	1		
9 – 12	0.683412	0.648557	0.833122	0.866553	1	
12 – 22	0.910879	0.886517	0.783909	0.814352	0.686893	1

Table 2 – *Descriptive statistics for iTraxx Europe series 5, 6, and 7. Panel A shows mean spreads for each series and tranche. The row "Total" refers to all three series. The row "Total*" refers to the shorter time series from 21.03.2006 to 30.06.2007 which excludes the high volatility phase of the crisis. Panel B shows spread volatilities. Panel C shows correlations among the index and the tranches. Panel D shows the same correlations but excluding the high volatility phase of the crisis.*

Figure 1 shows the implied correlation of all tranches over the three periods. At each day the correlations form a smile pattern. The implied correlations of equity and senior tranches are higher than those of the mezzanine tranches.

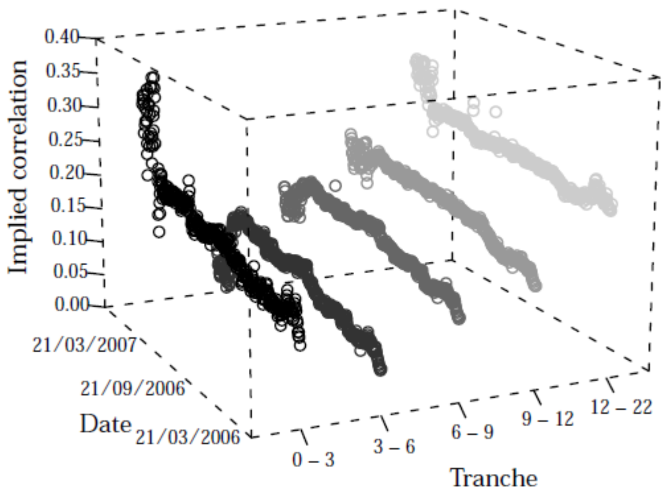


Figure 1 – *Time series of implied correlations for iTraxx Europe series 5 to 7.*

In terms of spreads this has the following meaning. A single average correlation seems to be able to correctly price the 0-3 and the 6-9 percent tranche. However, the 9-12 and the 12-22 percent tranches require a higher correlation and the 3-6 percent tranche requires a lower correlation. The market spread is too high and cannot be explained with the average correlation. This can be seen in the following graphs (**Figure 2**). They show model and market spread time series of the index and the tranches. Model spreads are plotted for different levels of pool asset correlation. The high volatility phase at the end of series 7 is omitted so as to improve the clarity of the graphs. However, it will be treated separately below.

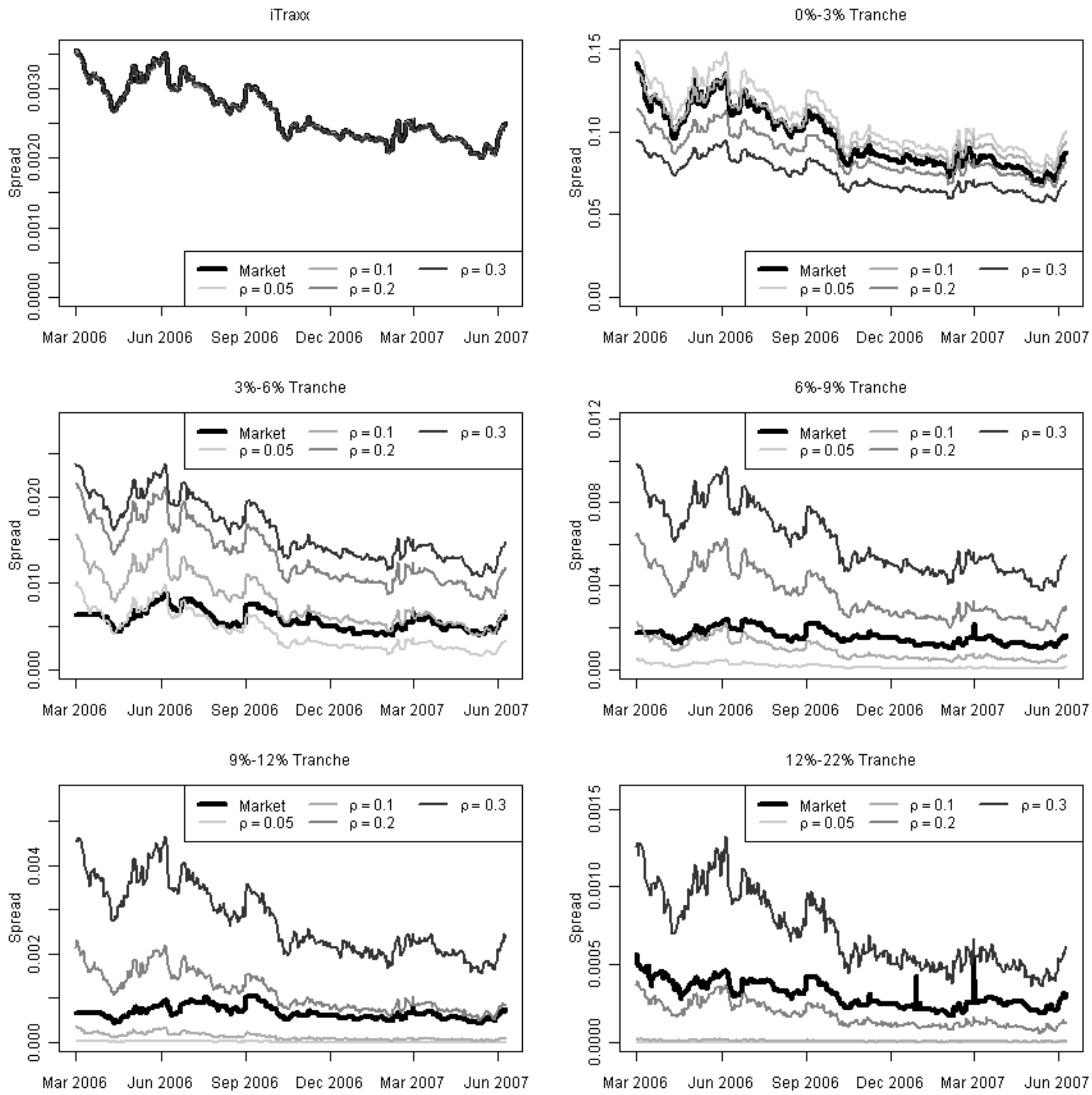


Figure 2 – Time series of observed spreads and Gaussian copula model spreads for four different levels of asset correlation. The first graph shows the index spread, the other five graphs show the five considered tranches. In each graph the market spreads are plotted with a bold black line. The thin lines are model spreads where a darker colour indicates a higher ρ. The high volatility phase from July 2007 to September 2007 is omitted.

The equity tranche spreads best match the model spreads with $\rho = 10\%$. The 3-6 percent tranche is closest to the model with $\rho = 5\%$ and $\rho = 10\%$, the 6-9 percent tranche is closest to the model with $\rho = 10\%$ and $\rho = 20\%$, the 9-12 percent tranche is closest to the model with $\rho = 20\%$, and finally, the 12-22 percent tranche is clearly between $\rho = 20\%$ and $\rho = 30\%$. This agrees with **Figure 1**.

In technical terms, the origin of the correlation smile seems obvious. The tails of the Gaussian copula model have too little probability mass. As already summarized in the first section, there are many suggestions on how to resolve this problem, such as skewed or fat-tailed distributions for $R_{i,T}$, non-Gaussian copulas with high tail dependence, a non-constant correlation parameter, correlated recoveries, and heterogeneous correlations or spreads. All these approaches have in common that the model is fitted to the observed tranche prices. Although this ad-hoc procedure allows resolving the correlation smile, it remains unclear which of these modifications are economically valid and thus lend themselves for pricing other tranches.

3 State Contingent Valuation

In order to achieve a stronger theoretical foundation and more modelling flexibility we build on a state contingent valuation approach similar to Coval et al. (2009). State-contingent valuation is a fundamental approach in financial economic theory and has its major roots in the works of Arrow (1964) and Debreu (1959). The basic principle is that each state of the economy can be associated with a price. These are called state prices. In order to determine the value of a security its cash-flows in each state are weighted with the corresponding state price. A typical proxy for the state is a market index and the state prices are derived from options on this index.

To implement this approach the Gaussian copula model is too simple. Instead we use the structural model of Merton (1974).

3.1 Asset and market return model

Let $V_{i,t}$ denote the asset value of firm *i* at time *t* which follows a geometric Brownian motion with drift μ_i and volatility σ_i, i.e.

$$\frac{dV_{i,t}}{V_{i,t}} = \mu_i \cdot dt + \sigma_i \cdot dW_{i,t}$$

where $W_{i,t}$ denotes a Wiener process. It is also assumed that for the firm value at time $t = 0$ $V_{i,0} > 0$ holds. The solution of this stochastic differential equation is

$$\ln \frac{V_{i,T}}{V_{i,0}} = \left(\mu_i - \frac{\sigma_i^2}{2} \right) \cdot T + \sigma_i \cdot \sqrt{T} \cdot W_{i,T} \ ,$$

where $\mu_i = r_f + \lambda_i$, i.e., the sum of the riskless rate r_f and the asset risk premium λ_i. $W_{i,T}$ is a standard normally distributed random variable.

To include correlations between asset values of different firms $W_{i,T}$ is assumed to be driven by both idiosyncratic as well as systematic risk

$$W_{i,T} = \beta_i \cdot \frac{\sigma_M}{\sigma_i} M_T + \frac{\sigma_{u,i}}{\sigma_i} U_{i,T}$$

M_T denotes the standard normally distributed market factor and $U_{i,T}$ denotes the idiosyncratic risk factor of firm *i* which is also standard normal. We assume that the random variables $U_{i,T}$ and M_T are independent for any *i*. β_i denotes the beta factor of firm *i*. σ_M and $\sigma_{u,i}$ are the volatilities of the market factor and of the idiosyncratic factor, respectively. Inserting this in the previous equation gives

$$\ln \frac{V_{i,T}}{V_{i,0}} = \left(\mu_i - \frac{\sigma_i^2}{2} \right) \cdot T + \beta_i \cdot \sigma_M \cdot \sqrt{T} \cdot M_T + \sigma_{u,i} \cdot \sqrt{T} \cdot U_{i,T}$$

This model is consistent with the Gaussian model introduced above. Standardization yields

$$R_{i,T} = \frac{\ln\left(V_{i,T} / V_{i,0}\right) - \left(\mu_i - \sigma_i^2 / 2\right) \cdot T}{\sigma_i \cdot \sqrt{T}} = \frac{\beta_i \cdot \sigma_M \cdot \sqrt{T} \cdot M_T + \sigma_{u,i} \cdot \sqrt{T} \cdot U_{i,T}}{\sqrt{\beta_i^2 \cdot \sigma_M^2 + \sigma_{u,i}^2} \cdot \sqrt{T}} =$$

$$= \sqrt{\rho_i} \cdot M_T + \sqrt{1 - \rho_i} \cdot U_{i,T}$$

where $\rho_i = \frac{\beta_i^2 \cdot \sigma_M^2}{\beta_i^2 \cdot \sigma_M^2 + \sigma_{u,i}^2}$.

Default of firm i occurs when the asset value falls short of the debt level K_i at maturity *T*. Thus, the (real-world) probability of default (PD) is given by

$$\begin{aligned} p_{i,T} &= P\left(V_{i,T} \le K_i \middle| V_{i,0}\right) \\ &= P\left(R_{i,T} \le c_{i,T}^P\right) \\ &= \Phi\left(c_{i,T}^P\right) \end{aligned}$$

where $c_{i,T}^P$ is a default threshold which is given by

$$c_{i,T}^{P} = \frac{ln \frac{K_i}{V_{i,0}} - (\mu_i - \sigma_i^2/2) \cdot T}{\sigma_i \sqrt{T}}$$

Assuming that the market value follows a geometric Brownian motion with drift μ_M and volatility σ_M the (log-) market return until time T is given by

$$\ln \frac{V_{M,T}}{V_{M,0}} = \left(\mu_M - \frac{\sigma_M^{\ 2}}{2} \right) \cdot T + \sigma_M \cdot \sqrt{T} \cdot M_T$$

where $\mu_M = r_f + \lambda_M - \delta_M$, λ_M is the equity market risk premium and δ_M denotes the market dividend yield. The market return, or equivalently its innovation M_T, represents the state space in our model.

3.2 State Price Density and Valuation

The present value of a tranche π_0^{tr} is determined by weighting state conditional tranche cash flows with state prices

$$\pi_0^{tr} = \int E(CF_T^{tr} \mid M_T) \cdot q(M_T) dM_T \tag{3}$$

$q(M_T)$ denotes the state price density[4]. $E(CF_T^{tr} \mid M_T)$ denotes the conditional expected tranche cash flow. It is based on the real-world (physical) loss measure and is directly related to the conditional expected tranche loss

$$E(CF_T^{tr} \mid M_T) = 1 - E(L_T^{tr} \mid M_T)$$

The state price density is determined based on option prices of an equity index that represents the state of the credit market. The exact procedure is due to Breeden and Litzenberger (1978). They show that state prices $q(K)$ can be determined as the second derivative of the European call price with respect to the strike price

$$q(K) = \frac{d^2 C\big(K, T, \sigma(K)\big)}{dK^2} \tag{4}$$

Here, K denotes the strike price, $\sigma(K)$ denotes the implied volatility, and $C\big(K, T, \sigma(K)\big)$ denotes the Black-Scholes price of a European call option. The implied volatility is that

[4] The state price density is the discounted risk neutral density.

volatility for which the Black-Scholes option price is equal to the market option price. The formulas for $q(K)$ and $q(M_T)$ can be found in Appendix 1.

Given a set of parameters including the asset volatility the Black-Scholes model yields option prices for any strike price. Conversely, if the market relies on a Black-Scholes valuation, the observed option-prices can be explained with the same volatility. Empirically, however, this does not hold true. When volatilities are implied from option prices for different strike prices a well known phenomenon called “volatility smile” or “volatility skew” can be observed. This denotes the fact that out-of-the-money and in-the-money options require higher Black Scholes volatilities to reach the market prices than at-the-money options. These differences in implied volatilities for different strike prices are partly explained by higher risk premia for adverse states of the market.

The second derivative in the Breeden-Litzenberger formula requires that $\sigma(K)$ be a twice differentiable function of the strike price. A common choice for such a fitting function is “tangens hyperbolicus”. We use the following parametric form to explain $\sigma(K)$:

$$\hat{\sigma}(K) = \beta_0 + \beta_1 \tanh\left(-\beta_2 \ln\left(K / V_{M,0}\right)\right)$$

We calibrate the parameters $\beta_0, \beta_1,$ and β_2 by minimizing the sum of squared differences between observed option prices and model prices over all available K, i.e.,

$$\arg\min_{\beta_0,\beta_1,\beta_2} \sum_{j=1}^{J} \left[C\left(K_j, T\right) - \hat{C}\left(K_j, T, \hat{\sigma}\left(K_j\right)\right)\right]^2$$

where j is the strike price index (running from unity to J) and $\hat{C}\left(K_j, T, \hat{\sigma}\left(K_j\right)\right)$ denotes the Black Scholes option price for strike price K_j, maturity T, and volatility $\hat{\sigma}\left(K_j\right)$. **Figure 3** shows the volatility smile and the fitted tangens hyperbolicus function on a specific day.

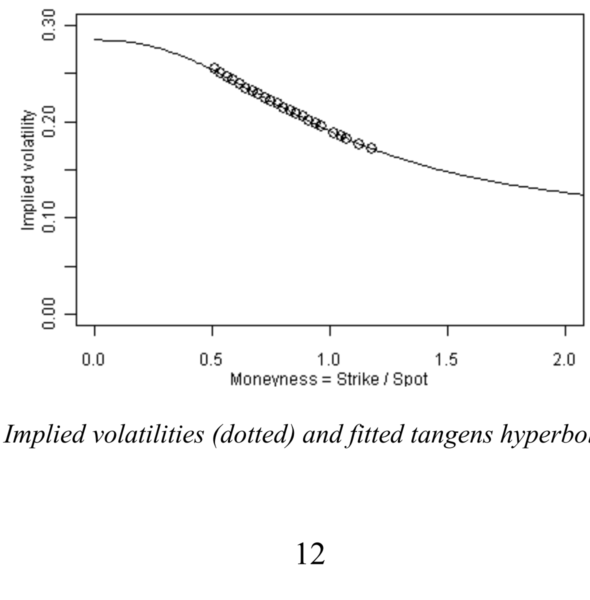


***Figure 3** – Implied volatilities (dotted) and fitted tangens hyperbolicus function.*

Finally, $q(K)$ in (4) and $q(M_T)$ in (3) are related via the above market index return equation

$$\frac{\ln\frac{K}{V_{M,0}}-(\mu_M-\sigma_M{}^2/2)\cdot T}{\sigma_M\sqrt{T}}=M_T$$

Beside $q(M_T)$ in (3) we need the conditional tranche cash flows, $E(CF_T^{tr}\mid M_T)$ or the conditional tranche expected losses, $E(L_T^{tr}\mid M_T)$, to determine the value of a tranche. Tranche losses (2) are based on pool losses (1). The pool is assumed to be homogeneous and has three parameters: the recovery rate *RR*, the asset correlation ρ and the default threshold c_T. The recovery rate and the pool asset correlation are typically fixed externally. The default threshold in the pool, c_T, is calibrated so that the model price for the pool equals the observed index price. The pool price is simply the expectation of the pool cash flows with respect to the state price density

$$\pi_0=\int\left(1-E(L_T\mid M_T)\right)\cdot q(M_T)dM_T$$

where

$$\begin{aligned}E(L_T\mid M_T)&=E\left(\mathbf{1}_{\{R_T\le c_T\}}\right)(1-RR)\\&=\Phi\left(\frac{c_T-\sqrt{\rho}\cdot M_T}{\sqrt{1-\rho}}\right)(1-RR)\end{aligned}$$

Thus, the default threshold is used to include credit market information in our model. However, contrary to the Gaussian copula valuation no tranche prices or tranche spreads are used to calibrate the model.

Finally, based on the present value of the tranche, π_0^{tr}, the tranche spread is given by

$$s^{tr}=-\frac{1}{T}\ln\left(\pi_0^{tr}\right)-r_f$$

3.3 Comparison with Standard Gaussian Copula Valuation

It is helpful at this point to compare the state-contingent valuation approach with the standard approach as described in Section 2. The market factor M_T in the latter is standard Gaussian. In the state contingent valuation approach the market factor is Gaussian if a constant volatility is assumed. In this case both approaches only differ in terms of their cash flow structure. The Gaussian copula model includes quarterly payments while the state price approach is static. Practically, this difference is less relevant. It is easy to see that when the Gaussian copula model refers to one period only and both models are calibrated to the same index spread, then they are equivalent. With the Gaussian copula model the full risk neutral shift is in the obligors' PD. In the state contingent pricing model part of the shift is in the PD and the other part is in the implied factor density.

This can be shown as follows. The pool price based on a state-contingent valuation approach is given by

$$\pi_0 = \int \left(1 - \Phi\left(\frac{c_T - \sqrt{\rho} \cdot M_T}{\sqrt{1-\rho}} \right) (1 - RR) \right) \cdot q(M_T) dM_T$$

With the assumption of a constant volatility we have $\sigma(K,T) = \sigma_M$. After transformation from K to M_T this yields

$$q(M_T) = \varphi\left(M_T - (r_f - \mu_M)\sqrt{T} / \sigma_M \right) \cdot \exp(-r_f \cdot T)$$

where $\varphi(M_T)$ denotes the standard Gaussian density. Hence, $q(M_T)$ is the density of a normal distribution with mean $(r_f - \mu_M)\sqrt{T} / \sigma_M$ and standard deviation one.

By a change of variables $\widetilde{M}_T = M_T - (r_f - \mu_M)\sqrt{T} / \sigma_M$ we obtain

$$\begin{aligned}
\pi_0 &= \int \left(1 - \Phi\left(\frac{c_T - \sqrt{\rho} \cdot \left(\widetilde{M}_T + (r_f - \mu_M)\sqrt{T} / \sigma_M \right)}{\sqrt{1-\rho}} \right) (1 - RR) \right) \\
&\quad \cdot \varphi(\widetilde{M}_T) \cdot \exp(-r_f \cdot T) d\widetilde{M}_T \\
&= \int \left(1 - \Phi\left(\frac{c_T^* - \sqrt{\rho} \cdot \widetilde{M}_T}{\sqrt{1-\rho}} \right) (1 - RR) \right) \cdot \varphi(\widetilde{M}_T) \cdot \exp(-r_f \cdot T) d\widetilde{M}_T
\end{aligned}$$

where $c_T^* = c_T - \sqrt{\rho}(r_f - \mu_M)\sqrt{T} / \sigma_M$ and $\widetilde{M}_T$ is a standard normal random variable. The last row is equivalent to the price based on the static Gaussian copula model.

If, on the other hand, the volatility skew is allowed for, the resulting density is non-standard and has typically a hump at the lower end. As a consequence, the factor density has a fat left tail. **Figure 4** shows an example. The bold curve is the risk neutral density resulting from the Breeden-Litzenberger procedure in which the volatility smile is allowed for. The curve has a pronounced hump at the lower end where the adverse market states are situated.

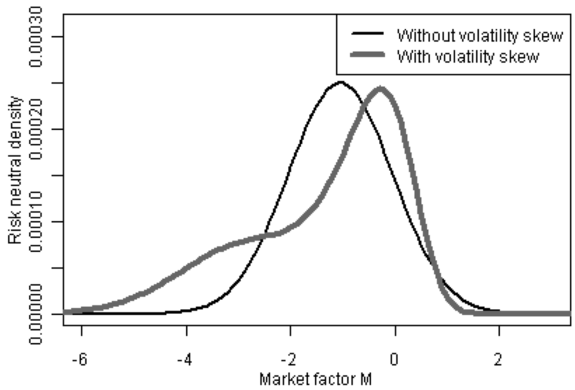


Figure 4 *– Risk neutral density without and with allowing for the volatility skew.*

By contrast, when the volatility skew is disregarded and the at-the-money volatility is assumed for all strike prices a Gaussian risk neutral density of the market factor is the result. This density has clearly less probability mass in the left tail. Thus, an asset the losses of which are concentrated in the left, adverse states, has a much lower value when the bold density is applied.

3.4 Empirical Results

We apply this pricing model now to the iTraxx Europe series 5, 6, and 7. In addition to CDS spreads and index spreads we use further data. The "Dow Jones Euro Stoxx 50", which comprises the 50 largest companies in the euro zone, acts as our market index for iTraxx Europe. There is a large overlap between iTraxx Europe and Euro Stoxx 50[5]. We estimate the annual (logarithmic) return of the price index on the history of the index since 1992.

To determine the state price density based on the Breeden-Litzenberger approach prices of five-year options (call and put) on this index are used[6]. We use 28 option prices for moneyness levels between 0.5 and 1.3[7]. The five-year euro area yield curve of AAA-rated government bonds is used for the daily riskless interest rate. The dividend yield is derived from daily quotes of five-year options on the Euro Stoxx index. The historical market return is 0.1270.

[5] For example, 44 of the 50 Euro Stoxx firms are in at least one of the three iTraxx series. Furthermore, 33 of the 50 Euro Stoxx firms are in all three series.

[6] We thank Bayerische Landesbank for the data access within our research cooperation.

[7] Moneyness is defined as $K/V_{M,0}$.

Figure 5 shows model and market spread time series of the index and the tranches. Model spreads are plotted for different levels of pool asset correlation.

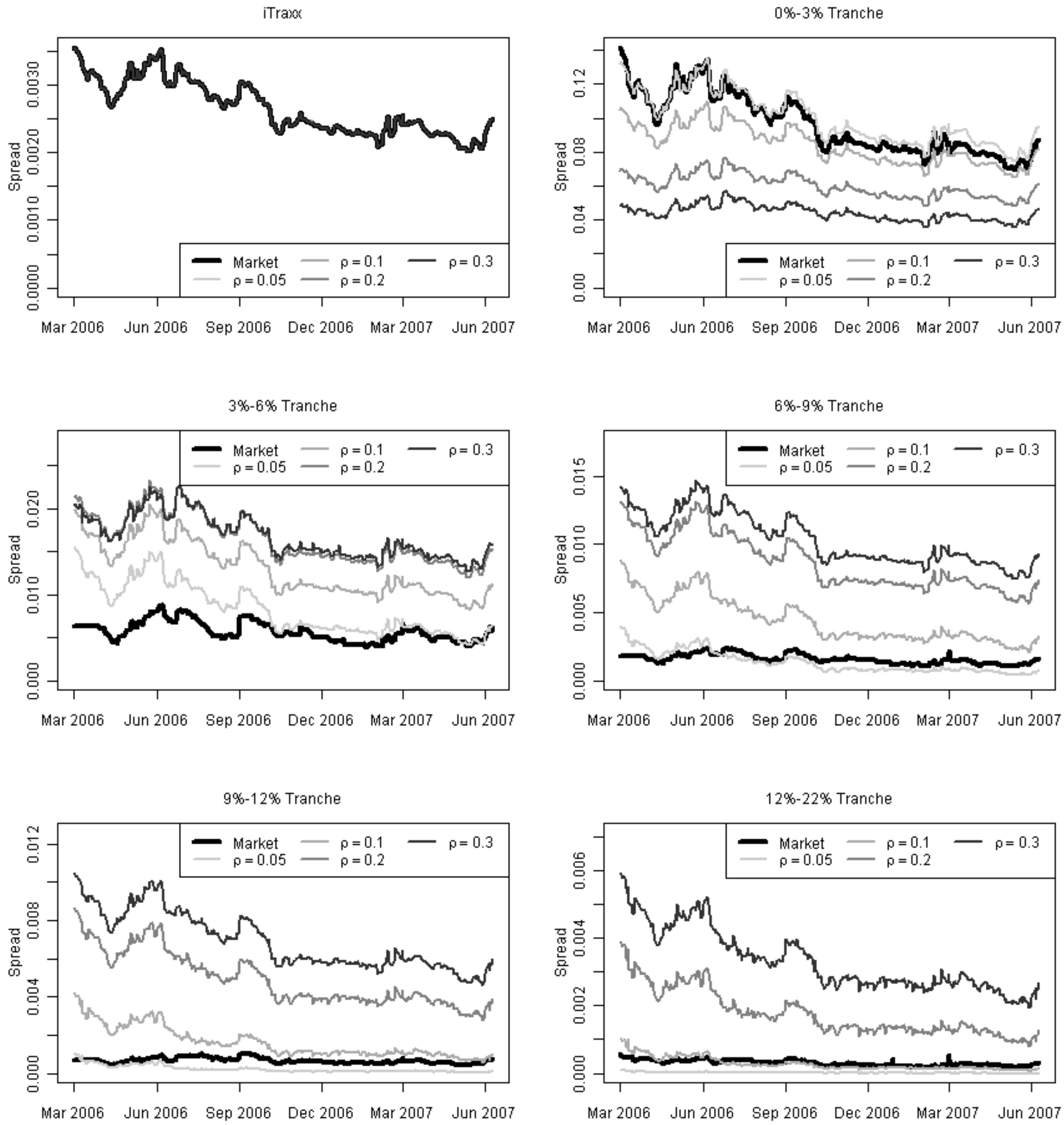


***Figure 5** – Time series of observed spreads and state-contingent valuation model spreads for four different levels of asset correlation. The first graph shows the index spread, the other five graphs show the five considered tranches. In each graph the market spreads are plotted with a bold black line. The thin lines are model spreads where a darker colour indicates a higher ρ. The high volatility phase from July 2007 to September 2007 is omitted.*

The equity tranche spreads are systematically underestimated for any asset correlation. The spreads of all other tranches are systematically overestimated for the 10, 20, and 30 percent asset correlation. This is clearly a result of the fat left tail of the state price density which implies significant discounts for all tranches with losses concentrated in the adverse states of the market factor.

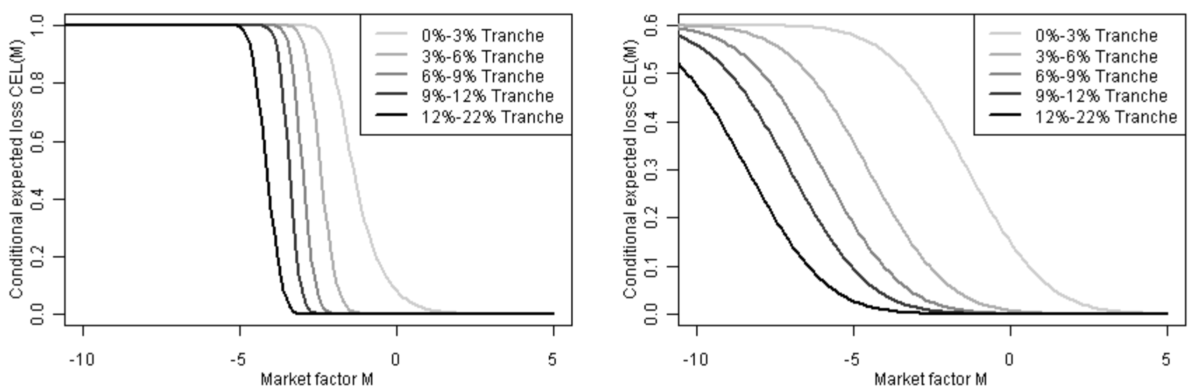
4 Tranche Correlation Adjustment

The state contingent pricing model of the last section includes two major components, the state price density and the expected conditional tranche loss. The result of the last two sections is that not allowing for the volatility skew and just pricing the correlation risk implies an underestimation of senior tranche spreads. By contrast, allowing for the skew implies a strong overestimation of all non-equity spreads.

With respect to the other pricing component there is a seeming contradiction. A series of studies (e.g. BIS 2005, BIS 2009) argue that investors make little difference between equally rated corporate bonds and structured securities. That is, they focus on rating information and do not allow for differences in the risk profiles. However, it is well documented that there are significant differences between corporate bonds and structured securities. The question arises whether this is price relevant.

4.1 Risk Adjustment

For illustration of the risk differences consider the following conditional expected loss profiles (**Figure 6**).



(a) Tranche   *(b) Bond*

Figure 6 *– Conditional expected loss profiles of the tranches as of 16.05.2006. The left graph (a) shows the tranche profiles. The right graph (b) shows the profiles of corporate bonds with equal expected loss as the corresponding tranches in (a) but with bond typical low correlation ρ = 0.20. Curves with the same greyscale in (a) and (b) have equal expected loss.*

The left panel shows the conditional expected loss of each iTraxx Europe tranche at a specific day (16.05.2006). The unconditional expected losses of these tranches at this date are given in **Table 3**.

Tranche	Expected Loss $E(L^{tr})$	Implicit Asset Correlation ρ
0 – 3	0.16453	0.59893
3 – 6	0.01304	0.85773
6 – 9	0.00227	0.90310
9 – 12	0.00051	0.92223
12 – 22	5.4191e-05	0.87496
Index	0.00541	0.20000

Table 3 *– Expected loss and implicit asset correlation of iTraxx Europe tranches as of 16.05.2006.*

The right panel in **Figure 6** shows the conditional expected loss of bonds with the same unconditional expected loss as the tranches. The conditional expected loss of a bond is given by

$$E(L_T \mid M_T) = \Phi\left(\frac{c_T - \sqrt{\rho} \cdot M_T}{\sqrt{1-\rho}}\right)(1 - RR) \tag{5}$$

For **Figure 6** the default threshold c_T in (5) is calibrated so that each tranche in the left panel and its corresponding bond curve in the right panel have the same expected loss, i.e., $E(E(L_T \mid M_T)) = E(L^{tr})$. The asset correlation is fixed at $\rho = 0.2$ which is equal to the asset correlation that is assumed for the CDS in the index pool. Finally, the recovery rate is fixed at the typical value of 0.4.

Although the graphs are similar, there are two major differences between tranches and bonds. First, the bonds have lower maximum loss. Second, the tranche curves are much steeper than the bond curves. In other words, they are more sensitive to changes of the systematic risk factor *M*. In the Gaussian single risk factor model for bonds the parameter driving factor sensitivity is ρ . A corporate bond has typically asset correlations ρ^{Bond} in the range of 0.15 to 0.30. The higher ρ , the steeper the conditional expected loss curve. Previous research has shown how to compare the sensitivity of corporate bonds with those of tranches (e.g. Moody's KMV (2008); Donhauser et al. (2010)). The basic idea is to calibrate the Gaussian single risk factor model (Formula (5)) so that it optimally fits the tranche expected loss curve. Hence, the calibrated rho is typically much higher than that of a corporate bond. The range of typical values depends on the seniority. While junior and senior tranches have correlations ρ^{tr} in the range of 0.5 to 0.7, those of mezzanine tranches are as high as 0.8 or 0.9. The implicit asset correlations of the iTraxx Europe tranches at 16.05.2006 are given in the last column of **Table 3**. They are clearly higher than those of the index.

The models studied so far are implicitly based on the high correlation of tranches. We implement the idea of a tranche with corporate bond asset correlation as follows. As shown above, the conditional expected loss of a tranche differs from that of a corporate bond in terms of the asset correlation ρ. Hence, we replace $E(L_T^{tr} \mid M_T)$ by the conditional expected loss of a bond

$$E(L_T^{adj} \mid M_T) = \Phi\left(\frac{c_T - \sqrt{\rho} \cdot M_T}{\sqrt{1-\rho}}\right)(1 - RR)$$

The first term on the right hand side is the conditional PD of the Gaussian single risk factor model as outlined in Section 2. The second term is the loss given default. For ρ the asset correlation of the bonds in the underlying pool is used. For example, with respect to Table 3, $\rho = 0.20$ is used for the equity tranche instead of $\rho = 0.5989$. To make the prices of these correlation adjusted bonds comparable to the original tranches, c_T has to be chosen so that

$$E\left(E(L_T^{adj} \mid M_T)\right) = E\left(E(L_T^{tr} \mid M_T)\right)$$

i.e., the (unconditional) expected loss of the tranche is maintained. Altogether, each tranche is expressed as a corporate bond with its typical lower correlation but with equal expected loss as the tranche.

4.2 Empirical Results

We apply the model using an adjusted tranche correlation of ρ. All other parameters are equal to those of the previous model. **Figure 7** shows the resulting model prices for different levels of pool asset correlation.

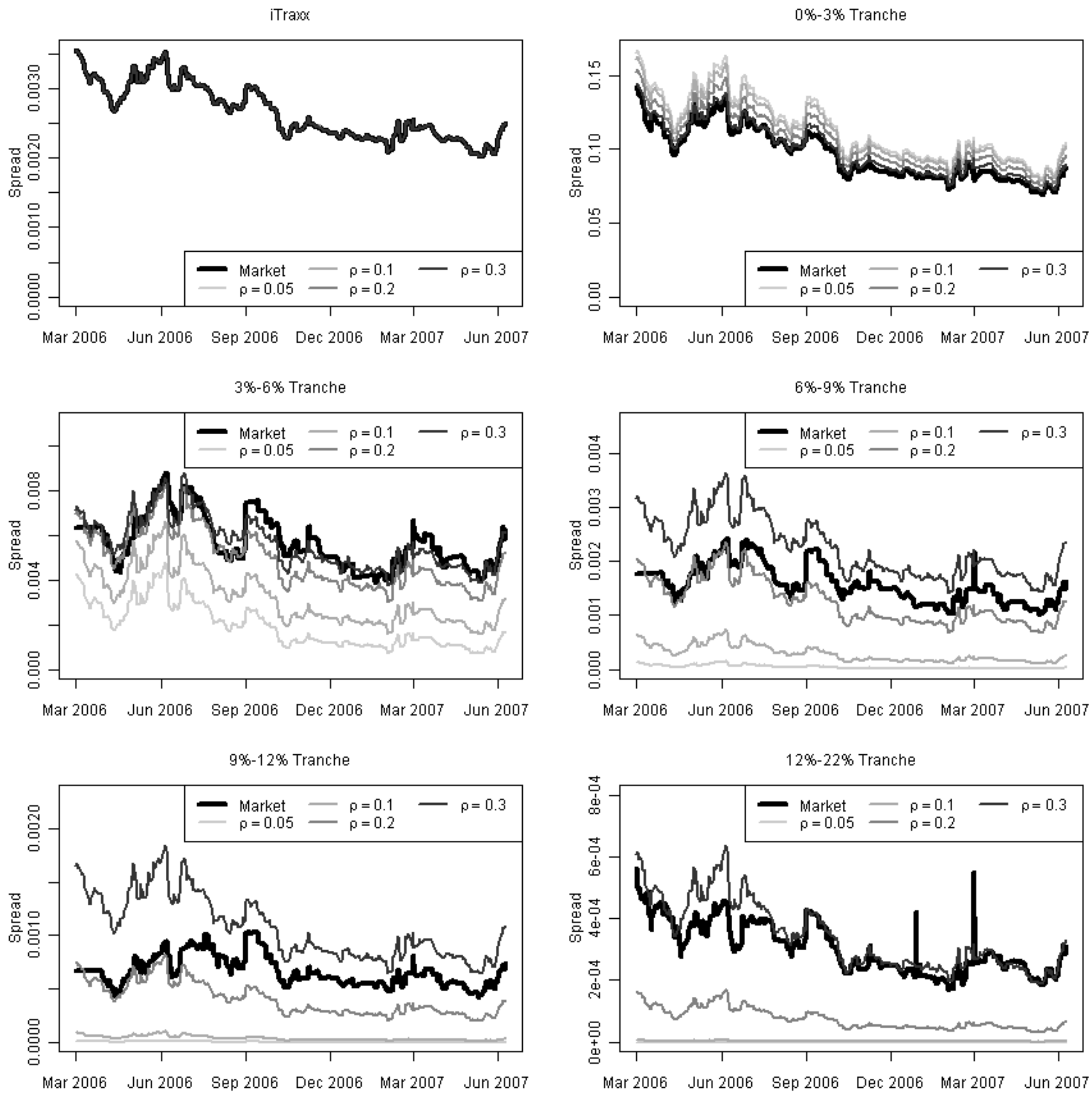


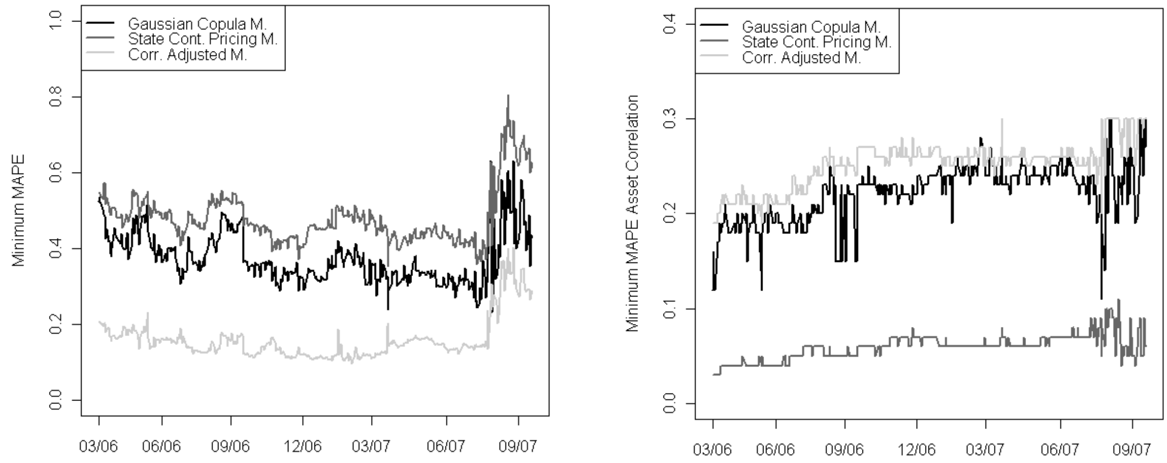
Figure 7 *– Time series of observed spreads and correlation adjusted state-contingent valuation model spreads for four different levels of asset correlation. The first graph shows the index spread, the other five graphs show the five considered tranches. In each graph the market spreads are plotted with a bold black line. The thin lines are model spreads where a darker colour indicates a higher ρ. The high volatility phase from July 2007 to September 2007 is omitted.*

For each tranche the market curve lies between the model curves with $\rho = 0.2$ and $\rho = 0.3$. Conversely, this means that the implied correlations based on this model always lie between 0.2 and 0.3. This model has clearly more explanatory power than the other two models which did not show a clear tendency. The Gaussian copula model underestimates equity and senior tranche spreads and overestimates mezzanine tranche spreads. The state-contingent pricing model always significantly underestimates equity spreads and overestimates the spreads of all other tranches. This is a result of the fat left tail of the risk neutral factor density as well as the high correlation of the tranches.

Subsequently, we measure the relative pricing error of the Gaussian Copula model, the state-contingent pricing model as well as the correlation adjusted model. For each day, each model, and each correlation in the range from 0.01 to 0.30 we calculate the mean absolute percentage error (MAPE)

$$MAPE_t = \frac{1}{5}\sum_{tr}\frac{\left|s_t^{tr} - \hat{s}_t^{tr}\right|}{\hat{s}_t^{tr}}$$

where s_t^{tr} denotes the model spread and $\hat{s}_t^{tr}$ denotes the observed spread on day t of tranche tr. Then, we determine the pool asset correlation which yields the lowest MAPE for each model on each day. This quantitative measure allows us to compare the models. Furthermore, we are able to judge the robustness of the models regarding exogenously given asset pool correlations. The results are shown in **Figure 8**.



(a) *(b)*

Figure 8 – Panel (a): *Time series of minimum MAPE for the Gaussian copula model, the state-contingent pricing model and the correlation adjusted state contingent pricing model. The graphs cover the series 5 to 7.* ***Panel (b):*** *Pool asset correlation associated with the minimum MAPE from Panel (a). The minimum asset correlation is chosen from a discrete set of values from 0.01 to 0.30 in 0.01 steps.*

The left graph shows the minimum MAPE over all admissible asset correlations on each day for each model over the three iTraxx Europe series. The right graph shows the corresponding asset correlations which minimize MAPE on each day for each model. The major finding is that the correlation adjusted model strongly dominates the other models in terms of relative percentage error. This applies for both the pre-crisis period as well as for the crisis period from June 2007 to September 2007. In the crisis all models are associated with higher errors. The right graph shows that the Gaussian copula model requires the largest range of asset correlations (from 0.12 to 0.27) to explain the market prices. This may be partly explained with the fact that the other two models obtain further market information from equity index options. The state contingent pricing model has optimal correlations in the range of 0.04 to 0.10. The correlation adjusted state contingent pricing model has optimal correlations in the range of 0.20 to 0.28. There is a wide range of estimated asset correlations in empirical studies. Asset correlations derived from default data are usually below 0.10 while asset correlations implied from equity price correlations are usually significantly higher between 0.10 and 0.30 (see Moody's 2008). Hence, we estimated asset correlations from the spreads of the iTraxx Europe members[8]. **Table 4** shows the average estimated asset correlation for each series. They are in fairly good agreement with the range of optimal correlations of the correlation adjusted model (0.20 to 0.30) but not with range of low correlations of the state contingent pricing model (0.02 to 0.10). This clearly supports the validity of the correlation adjusted model. With respect to the increased estimated correlations in the crisis period note that in Figure 8b the curve of the correlation adjusted model reaches the upper bound.

iTraxx	Period	Mean Empirical Asset Correlation
Series 5	21.03.2006 - 20.09.2006	0.278
Series 6	21.09.2006 - 20.03.2007	0.261
Series 7*	21.03.2007 - 30.06.2007	0.329
Series 7	21.03.2007 - 20.09.2007	0.555

Table 4 *– Empirical asset correlations for iTraxx Europe series 5 to 7. The row Series 7* is shorter and does not include the high volatility phase of the crisis. Mean empirical asset correlations are averages over all estimated pairwise correlations of the 125 names in the index.*

This indicates that even correlations higher than 0.30 would be optimal which is again in agreement with the estimated mean correlation of 0.555.

[8] See Tarashev and Zhou 2009 for a description of the exact procedure.

5 Conclusion

In this article we provide an economic explanation for the correlation smile phenomenon. We have shown that market prices of index tranches can be explained very well under the assumption that down-side risk premia are relevant but the increased systematic risk sensitivity of tranched securities is irrelevant. The relevance and robustness of our results is enhanced by the fact that we do not need to fit the model to tranche prices. The only calibration occurs with respect to the index spread level. The robustness is further supported by the fact that we do not rely on a single exogenous pool correlation estimate but rather examine a whole range of possible levels.

Our analysis also offers more insight into the risk components of structured debt and the valuation determinants. The comparison of the standard Gaussian copula valuation method and with the state contingent valuation method reveals explicitly that the former implicitly uses the true (high) tranche correlation but disregards extra down-side risk premia. This is due to the assumption of a Gaussian distribution for the market factor without a fat left tail. We have also shown that allowing for both down-side risk premia as well as the true tranche correlation results in a strong overestimation of the market prices unless extremely low asset correlations are assumed for the index.

Appendices

Appendix 1

The risk neutral density with respect to K is given by

$$
\begin{aligned}
q^{RND}(K) &= \frac{\partial^2 C}{\partial K^2} + \frac{\partial \sigma(K,T)}{\partial K} \cdot \left(2 \cdot \left(\frac{\partial^2 C}{\partial K \partial \sigma(K,T)} \right) + \frac{\partial^2 C}{\partial \sigma(K,T)^2} \cdot \left(\frac{\partial \sigma(K,T)}{\partial K} \right) \right) + \\
&\quad + \frac{\partial^2 \sigma(K,T)}{\partial K^2} \cdot \frac{\partial^2 C}{\partial \sigma(K,T)} \\
&= \varphi(d_2(K)) \cdot \left[\frac{1}{K \cdot \sigma(K,T)\sqrt{T}} + \frac{2 \cdot d_1(K)}{\sigma(K,T)} \cdot \frac{\partial \sigma(K,T)}{\partial K} + \right. \\
&\quad + \frac{K \cdot d_1(K) \cdot d_2(K) \cdot \sqrt{T}}{\sigma(K,T)} \cdot \left(\frac{\partial \sigma(K,T)}{\partial K} \right)^2 + \\
&\quad \left. + K \cdot \sqrt{T} \cdot \frac{\partial^2 \sigma(K,T)}{\partial K^2} \right]
\end{aligned}
$$

where

$$
\begin{aligned}
d_1(K) &= \frac{\ln(V_{M,0}/K) + \left(r_f + 0.5 \cdot \sigma(K,T)^2\right) \cdot T}{\sigma(K,T)\sqrt{T}} \\
d_2(K) &= d_1(K) - \sigma(K,T)\sqrt{T}
\end{aligned}
$$

The state price density is

$$
q(K) = q^{RND}(K) \cdot e^{-r_f T}
$$

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